A New Approach to Affine Parametric Quadratic Inverse Eigenvalue Problem

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Inverse Eigenvalue Problem

$$P(\lambda) = \lambda^2 M + \lambda C + K$$

$$C = C(\alpha) = C_0 + \sum_{i=1}^{n} \alpha_i C_i, \;\; K = K(\alpha) = K_0 + \sum_{i=1}^{n} \beta_i K_i,$$

$$(M, C(\alpha), K(\beta))$$

$$\alpha, \beta \in \mathsf{R}^n$$

$$C_i = C_i^T \in \mathsf{R}^{n \times n}, K_i = K_i^T \in \mathsf{R}^{n \times n}, \;\; M = M^T \in \mathsf{R}^{n \times n}$$

$$\{\lambda_1(\alpha, \beta), ..., \lambda_{2n}(\alpha, \beta)\}$$

Problem 1. Given a set with distinct entries $\{\lambda_1^*, ..., \lambda_{2n}^*\}$, closed under complex conjugation, find $(\alpha, \beta) \in \mathsf{R}^{2n}$, such that $\{\lambda_1^*, ..., \lambda_{2n}^*\}$ are eigenvalues of the pencil $(M, C(\alpha), K(\beta))$.

We will denote $\{\lambda_1^*, ..., \lambda_{2n}^*\}$ the target set of eigenvalues.

Newton's Method

Theorem *Let* (α^*, β^*) *be a solution to Problem 1 then there exist a neighborhood of* (α^*, β^*) *which contains no singular points. These are the points where the pencil has multiple eigenvalues.*

Corollary *There is a neighborhood of* (α^*, β^*) *where* $\lambda_i(\alpha, \beta)$ *are distinct and are differentiable functions.*

$$f(\alpha, \beta) = \begin{pmatrix} \lambda_{\sigma(1)}(\alpha, \beta) - \lambda_1^* \\ \vdots \\ \lambda_{\sigma(2n)}(\alpha, \beta) - \lambda_{2n}^* \end{pmatrix} = 0.$$

Where σ *is the permutation which minimizes*

$$\sum_{j=1}^{2n} |\lambda_{\sigma_i(j)}(\alpha, \beta) - \lambda_j^*|$$

among all possible permutations of the list of eigenvalues $\lambda_i(\alpha, \beta)$*.*

$$\sigma \in \arg \min_{\sigma_1, \ldots, \sigma_{n!}} \sum_{j=1}^{2n} |\lambda_{\sigma_i(j)}(\alpha, \beta) - \lambda_j^*|$$

Newton's Step

$$J(\alpha^i, \beta^i) \begin{pmatrix} \alpha^{i+1} - \alpha^i \\ \beta^{i+1} - \beta^i \end{pmatrix} = -f(\alpha^i, \beta^i)$$

$$J_{ik} = \begin{cases} \dfrac{\partial \lambda_i}{\partial \alpha_k} = -\dfrac{\lambda_i x_i^T C_k x_i}{x_i^T (2\lambda_i M + C) x_i}, & k = 1, \ldots, n \\ \dfrac{\partial \lambda_i}{\partial \beta_{k-n}} = -\dfrac{x_i^T K_{k-n} x_i}{x_i^T (2\lambda_i M + C) x_i}, & k = n+1, \ldots, 2n \end{cases}$$

Newton's Algorithm

Algorithm 1 Newton's Method

INPUT: λ^*, (α_0, β_0), tolerance - ϵ

OUTPUT: (α^*, β^*)

```
for i = 0,1,... do
    Find eigenvalues and eigenvectors of (M, C(α_i), K(β_i)).
    Solve the minimization combinatorics problem, compute σ
    (can be done by Hungarian Method in O(n^3))
    Compute f(α_i, β_i)
    Form J(α^i, β^i) and find (α^{i+1}, β^{i+1})
    Stop if ||(α^{i+1}, β^{i+1}) − (α^i, β^i)|| < ε
end for
```

Convergence

Conditions for quadratic convergence are satisfied:

- f is differentiable and J is Lipschitz continuous in a neighborhood of (α^*, β^*)
- $J(\alpha^*, \beta^*)$ is nonsingular

Alternative Projections

For given X, Λ^*, M we can compute C, K, so that

$$MX(\Lambda^*)^2 + CX\Lambda^* + KX = 0$$

$$V = \begin{pmatrix} X \\ X\Lambda^* \end{pmatrix}^{-1} = (V_1, V_2), \; V_1, V_2 \in \mathsf{R}^{2n\times n}$$

$$(C, K) = (-MX(\Lambda^*)^2 V_2, \; -MX(\Lambda^*)^2 V_1).$$

$$\mathcal{L} = \{(C,K) \in \mathsf{R}^{2n^2} \, |(C,K) = (-MX(\Lambda^*)^2 V_2, -MX(\Lambda^*)^2 V_1), \text{ for some matrix } X \text{ s.t } ||x_i|| = 1\}$$

and

$$\mathcal{A} = \{(C,K) \in \mathsf{R}^{2n^2} \mid C = C(\alpha) = C_0 + \sum_{i=1}^{n} \alpha_i C_i, \; K = K(\alpha) = K_0 + \sum_{i=1}^{n} \beta_i K_i\}.$$

Problem 2. Find $(\alpha, \beta) \in \mathsf{R}^{2n}$ such that $(M, C(\alpha), K(\beta)) \in \mathcal{L} \cap \mathcal{A}$.

Convergence

Theorem A. *Let* C_1, C_2 *be closed convex sets in a finitely dimensional Hilbert space* H, $C_1 \bigcap C_1 \neq \emptyset$ *and let* P_{C_1} *and* P_{C_2} *denote projection operators onto* C_1 *and* C_2 *correspondingly. Then*

$$\lim_{n \to \infty} (P_{C_1} P_{C_2})^n = \lim_{n \to \infty} (P_{C_2} P_{C_1})^n = P_{C_1 \cap C_2}$$

Note, both of the sets are closed, however the set $\mathcal{L}$ is nonconvex. Thus, alternating projections might not converge. However, alternating projection never increases the distance between successive iterates.

Theorem B. *Let* C_1, C_2 *be closed sets in a finitely dimensional Hilbert space* H, $C_1 \bigcap C_1 \neq \emptyset$ *and let* $y \in C_2$. *If*

$$x_1 = P_{C_1}(y), \; y_1 = P_{C_2}(x_1), \; x_2 = P_{C_1}(y_1),$$

then

$$||x_2 - y_1|| \leq ||x_1 - y_1|| \leq ||x_1 - y||.$$

Corrolary. *For any given* $x_0 \in H$, $\{(P_{C_1} P_{C_2})^n (x_0)\}_{n=0}^{\infty}$ *is a nondecreasing sequence.*

Projections onto L

$$< (C_1, K_1), (C_2, K_2) >= trace(C_1^T C_2 + K_1^T K_2)$$

Let $(C, K) \in \mathsf{R}^{2n^2}$ and $(X_\sigma, \Lambda_\sigma)$ be the eigenpair of the quadratic pencil (M, C, K). Define $V = (V_1, V_2)$ as

$$V = \begin{pmatrix} X_\sigma \\ X_\sigma \Lambda^* \end{pmatrix}^{-1}$$

Then

$$P_{\mathcal{L}}(C, K) = (-MX_\sigma(\Lambda^*)^2 V_2, -MX_\sigma(\Lambda^*)^2 V_1) = (\hat{C}, \hat{K}) \quad (1)$$

where σ is the permutation which solves the linear assignment problem , rows of X_σ is arranged is such a way so that i^{th} column of X_σ, $x_{\sigma(i)}$ is the eigenvector which corresponds to eigenvalue $\lambda_{\sigma(i)}$.

$$P_{\mathcal{L}}(C, K) \in \arg \min_{(X,Y) \in \mathcal{L}} ||X - C|| + ||Y - K||$$

Projection onto A

Let $P_{\mathcal{A}}(C, K) = (C(\hat{\alpha}), K(\hat{\beta}))$, then coefficients $(\hat{\alpha}, \hat{\beta})$ could be found as solutions of the following linear systems

$$A_1 \hat{\alpha} = b_1,\ A_2 \hat{\beta} = b_2,$$

where $(A_1)_{ij} = trace(C_i^T C_j)$, $(A_2)_{ij} = trace(K_i^T K_j)$,
$(b_1)_i = trace((C - C_0)^T C_i)$, $(b_2)_i = trace((K - K_0)^T K_i)$.

Alternative Projections Algorithm

Algorithm 1 Alternative Projections Method

INPUT: λ^*, (α^0, β^0), ϵ

OUTPUT: $(\hat{\alpha}, \hat{\beta})$

```
for i = 0,1,... do
    Form (C(α_i), K(β_i))
    Compute eigenvalues-eigenvectors of (M, C(α_i), K(β_i)) and compute σ
    Form matrix X_σ
    Compute (Ĉ, K̂), projection of (C(α_i), K(β_i)) onto ℒ
    Compute P_𝒜(C, K), (α_{i+1}, β_{i+1})
    Stop if ||(α^{i+1}, β^{i+1}) − (α^i, β^i)|| < ε
end for
```

Modification

Eigenvectors can by updated by one stef of inverse iteration.

$$\begin{pmatrix} 0 & I \\ K & C \end{pmatrix} \begin{pmatrix} u_j \\ z_j \end{pmatrix} = \lambda_j^* \begin{pmatrix} I & 0 \\ 0 & -M \end{pmatrix} \begin{pmatrix} x^i_{\sigma(j)} \\ \lambda_j^* x^i_{\sigma(j)} \end{pmatrix}, \ i = 1, ..., 2n.$$

We then define

$$x^{i+1}_{\sigma(j)} = \frac{u^j}{||u^j||}$$

which determines the new matrix X_σ^{i+1}. Vector u_j can be obtained as a solution of an $n \times n$ linear system

$$({\lambda_j^*}^2 M + \lambda_j^* C + K) u_j = (C + 2\lambda_j^* M) x^i_{\sigma(j)}$$

Algorithm

Algorithm 1 Alternative Projections - like Method

INPUT: λ^*, (α^0, β^0), ϵ

OUTPUT: $(\hat{\alpha}, \hat{\beta})$

1: Form $(C(\alpha_0), K(\beta_0))$

2: Compute eigenvalues-eigenvectors of $(M, C(\alpha_0), K(\beta_0))$ and compute σ

3: Form matrix $X_\sigma^{(0)} = X_\sigma(\alpha_0, \beta_0)$

4: **for** i = 0,1,... **do**

5: Compute $(\hat{C}, \hat{K})$, quasi-projection of $(C(\alpha_i), K(\beta_i))$ onto $\mathcal{L}$

6: Compute $(\alpha_{i+1}, \beta_{i+1})$

7: Stop if $||(\alpha^{i+1}, \beta^{i+1}) - (\alpha^i, \beta^i)|| < \epsilon$

8: Form $(C, K) = (C(\alpha_{i+1}), K(\beta_{i+1}))$

9: Solve $2n$ linear systems

$$((\lambda_j^*)^2 M + \lambda_j^* C + K) u_j = (C + 2\lambda_j^* M) x_{\sigma(j)}^i, \ j = 1, ..., 2n$$

and compute

$$x_{\sigma(j)}^{i+1} = \frac{u^j}{||u^j||}$$

10: **end for**

Hybrid Method

Algorithm 1 Hybrid Method

INPUT: λ^*, (α^0, β^0), ϵ_1, ϵ_2

OUTPUT: (α^*, β^*)

while $||(\alpha^{i+1}, \beta^{i+1}) - (\alpha^i, \beta^i)|| < \epsilon_2$ **do**

    $(\alpha^{i+1}, \beta^{i+1}) = c((P_{\mathcal{A}} P_{\mathcal{L}})(C(\alpha^i), K(\beta^i)))$

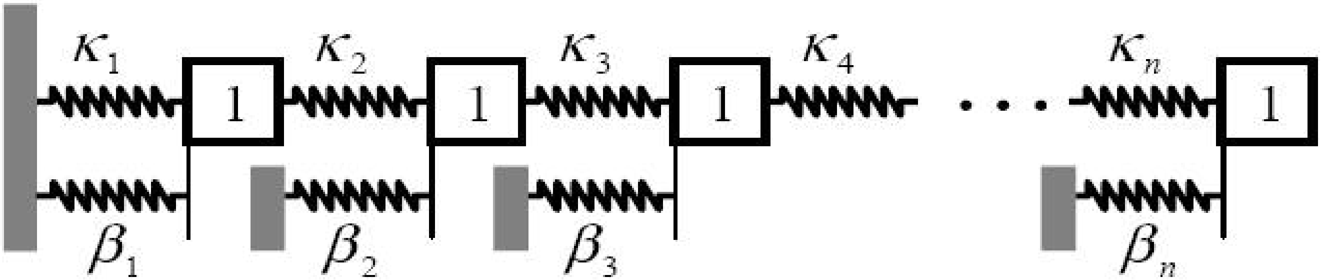
end while

while $||(\alpha^{i+1}, \beta^{i+1}) - (\alpha^i, \beta^i)|| < \epsilon_1$ **do**

    Form $J(\alpha^i, \beta^i)$ and compute $(\alpha^{i+1}, \beta^{i+1})$

end while

Example



$$
K_0 = \begin{pmatrix} \kappa_1+\kappa_2 & -\kappa_2 & & & \\ -\kappa_2 & \kappa_2+\kappa_3 & -\kappa_3 & & \\ & -\kappa_3 & \kappa_3+\kappa_4 & -\kappa_4 & \\ & & \ddots & \ddots & \ddots \\ & & & -\kappa_n & \kappa_n \end{pmatrix}
$$

$$
\begin{aligned}
&\alpha_k = \kappa_k, \;\; k = 1, 2, \ldots, n, \\
&A_0 = O, \;\; A_1 = e_1 e_1^T, \\
&A_k = (e_{k-1} - e_k)(e_{k-1} - e_k)^T, \;\; k = 2, 3, \ldots, n
\end{aligned}
$$

Numerical Experiment

$$\alpha_0 = (1,1,1),\ \beta_0 = (1,1,1)$$

$$\lambda^* = \{-0.0271 \pm i1.0108, -0.0177 \pm i0.6724, -0.0023 \pm i0.2658\}$$

Solution of Modified Alternatig Projection Method:

$$\alpha_{AP} = (0.0332, 0.0134, 0.0169),\ \beta_{AP} = (0.7188, 0.2193, 0.1915)$$

$$\lambda(\alpha_{AP}, \beta_{AP}) = \{-0.0271 \pm i1.0110, -0.0176 \pm i0.6724, -0.0021 \pm i0.2554\}$$

Locally Unique Solution of Newton'n Method:

$$\alpha_N = (0.0139, 0.0203, 0.0199),\ \beta_N = (0.6038, 0.2722, 0.1988)$$

$$||\lambda^* - \lambda(\alpha_N, \beta_N)|| = 1.29 \times 10^{-9}$$