

# Scientific Computing: An Introductory Survey

## Chapter 6 – Optimization

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### Optimization

- Given function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$ , and set  $S \subseteq \mathbb{R}^n$ , find  $x^* \in S$  such that  $f(x^*) \leq f(x)$  for all  $x \in S$
- $x^*$  is called *minimizer* or *minimum* of  $f$
- It suffices to consider only minimization, since maximum of  $f$  is minimum of  $-f$
- Objective* function  $f$  is usually differentiable, and may be linear or nonlinear
- Constraint* set  $S$  is defined by system of equations and inequalities, which may be linear or nonlinear
- Points  $x \in S$  are called *feasible* points
- If  $S = \mathbb{R}^n$ , problem is *unconstrained*



### Examples: Optimization Problems

- Minimize weight of structure subject to constraint on its strength, or maximize its strength subject to constraint on its weight
- Minimize cost of diet subject to nutritional constraints
- Minimize surface area of cylinder subject to constraint on its volume:

$$\min_{x_1, x_2} f(x_1, x_2) = 2\pi x_1(x_1 + x_2)$$

$$\text{subject to } g(x_1, x_2) = \pi x_1^2 x_2 - V = 0$$

where  $x_1$  and  $x_2$  are radius and height of cylinder, and  $V$  is required volume



### Global Optimization

- Finding, or even verifying, global minimum is difficult, in general
- Most optimization methods are designed to find local minimum, which may or may not be global minimum
- If global minimum is desired, one can try several widely separated starting points and see if all produce same result
- For some problems, such as linear programming, global optimization is more tractable



### Outline

- Optimization Problems
- One-Dimensional Optimization
- Multi-Dimensional Optimization



### Optimization Problems

- General continuous optimization problem:

$$\min f(x) \quad \text{subject to } g(x) = 0 \quad \text{and} \quad h(x) \leq 0$$

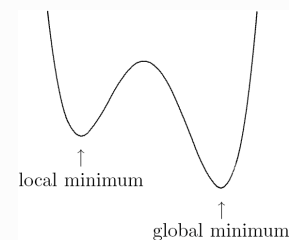
where  $f: \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $g: \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $h: \mathbb{R}^n \rightarrow \mathbb{R}^p$

- Linear programming*:  $f$ ,  $g$ , and  $h$  are all linear
- Nonlinear programming*: at least one of  $f$ ,  $g$ , and  $h$  is nonlinear



### Local vs Global Optimization

- $x^* \in S$  is *global minimum* if  $f(x^*) \leq f(x)$  for all  $x \in S$
- $x^* \in S$  is *local minimum* if  $f(x^*) \leq f(x)$  for all feasible  $x$  in some neighborhood of  $x^*$



### Existence of Minimum

- If  $f$  is continuous on *closed* and *bounded* set  $S \subseteq \mathbb{R}^n$ , then  $f$  has global minimum on  $S$
- If  $S$  is not closed or is unbounded, then  $f$  may have no local or global minimum on  $S$
- Continuous function  $f$  on unbounded set  $S \subseteq \mathbb{R}^n$  is *coercive* if
 
$$\lim_{\|x\| \rightarrow \infty} f(x) = +\infty$$
 i.e.,  $f(x)$  must be large whenever  $\|x\|$  is large
- If  $f$  is coercive on closed, unbounded set  $S \subseteq \mathbb{R}^n$ , then  $f$  has global minimum on  $S$



## Level Sets

- **Level set** for function  $f: S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  is set of all points in  $S$  for which  $f$  has some given constant value
- For given  $\gamma \in \mathbb{R}$ , **sublevel set** is
 
$$L_\gamma = \{x \in S : f(x) \leq \gamma\}$$
- If continuous function  $f$  on  $S \subseteq \mathbb{R}^n$  has nonempty sublevel set that is closed and bounded, then  $f$  has global minimum on  $S$
- If  $S$  is unbounded, then  $f$  is coercive on  $S$  if, and only if, all of its sublevel sets are bounded



## First-Order Optimality Condition

- For function of one variable, one can find extremum by differentiating function and setting derivative to zero
- Generalization to function of  $n$  variables is to find **critical point**, i.e., solution of nonlinear system

$$\nabla f(x) = \mathbf{0}$$

where  $\nabla f(x)$  is **gradient** vector of  $f$ , whose  $i$ th component is  $\partial f(x)/\partial x_i$

- For continuously differentiable  $f: S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ , any interior point  $x^*$  of  $S$  at which  $f$  has local minimum must be critical point of  $f$
- But not all critical points are minima: they can also be maxima or saddle points



## Constrained Optimality

- If problem is constrained, only **feasible** directions are relevant
- For equality-constrained problem

$$\min f(x) \quad \text{subject to} \quad g(x) = \mathbf{0}$$

where  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  and  $g: \mathbb{R}^n \rightarrow \mathbb{R}^m$ , with  $m \leq n$ , necessary condition for feasible point  $x^*$  to be solution is that negative gradient of  $f$  lie in space spanned by constraint normals,

$$-\nabla f(x^*) = J_g^T(x^*)\lambda$$

where  $J_g$  is Jacobian matrix of  $g$ , and  $\lambda$  is vector of **Lagrange multipliers**

- This condition says we cannot reduce objective function without violating constraints



## Constrained Optimality, continued

- Together, necessary condition and feasibility imply critical point of Lagrangian function,

$$\nabla \mathcal{L}(x, \lambda) = \begin{bmatrix} \nabla f(x) + J_g^T(x)\lambda \\ g(x) \end{bmatrix} = \mathbf{0}$$

- Hessian of Lagrangian is symmetric, but not positive definite, so critical point of  $\mathcal{L}$  is saddle point rather than minimum or maximum
- Critical point  $(x^*, \lambda^*)$  of  $\mathcal{L}$  is constrained minimum of  $f$  if  $B(x^*, \lambda^*)$  is positive definite on **null space** of  $J_g(x^*)$
- If columns of  $Z$  form basis for null space, then test **projected Hessian**  $Z^T B Z$  for positive definiteness



## Uniqueness of Minimum

- Set  $S \subseteq \mathbb{R}^n$  is **convex** if it contains line segment between any two of its points
- Function  $f: S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  is **convex** on convex set  $S$  if its graph along any line segment in  $S$  lies **on or below** chord connecting function values at endpoints of segment
- Any local minimum of convex function  $f$  on convex set  $S \subseteq \mathbb{R}^n$  is global minimum of  $f$  on  $S$
- Any local minimum of **strictly** convex function  $f$  on convex set  $S \subseteq \mathbb{R}^n$  is **unique** global minimum of  $f$  on  $S$



## Second-Order Optimality Condition

- For twice continuously differentiable  $f: S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ , we can distinguish among critical points by considering **Hessian matrix**  $H_f(x)$  defined by

$$\{H_f(x)\}_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}$$

which is symmetric

- At critical point  $x^*$ , if  $H_f(x^*)$  is
  - positive definite, then  $x^*$  is minimum of  $f$
  - negative definite, then  $x^*$  is maximum of  $f$
  - indefinite, then  $x^*$  is saddle point of  $f$
  - singular, then various pathological situations are possible



## Constrained Optimality, continued

- **Lagrangian function**  $\mathcal{L}: \mathbb{R}^{n+m} \rightarrow \mathbb{R}$ , is defined by

$$\mathcal{L}(x, \lambda) = f(x) + \lambda^T g(x)$$

- Its gradient is given by

$$\nabla \mathcal{L}(x, \lambda) = \begin{bmatrix} \nabla f(x) + J_g^T(x)\lambda \\ g(x) \end{bmatrix}$$

- Its Hessian is given by

$$H_{\mathcal{L}}(x, \lambda) = \begin{bmatrix} B(x, \lambda) & J_g^T(x) \\ J_g(x) & O \end{bmatrix}$$

where

$$B(x, \lambda) = H_f(x) + \sum_{i=1}^m \lambda_i H_{g_i}(x)$$



## Constrained Optimality, continued

- If inequalities are present, then KKT optimality conditions also require nonnegativity of Lagrange multipliers corresponding to inequalities, and complementarity condition



## Sensitivity and Conditioning

- Function minimization and equation solving are closely related problems, but their sensitivities differ
- In one dimension, absolute condition number of root  $x^*$  of equation  $f(x) = 0$  is  $1/|f'(x^*)|$ , so if  $|f(\hat{x})| \leq \epsilon$ , then  $|\hat{x} - x^*|$  may be as large as  $\epsilon/|f'(x^*)|$
- For minimizing  $f$ , Taylor series expansion

$$\begin{aligned} f(\hat{x}) &= f(x^* + h) \\ &= f(x^*) + f'(x^*)h + \frac{1}{2}f''(x^*)h^2 + \mathcal{O}(h^3) \end{aligned}$$

shows that, since  $f'(x^*) = 0$ , if  $|f(\hat{x}) - f(x^*)| \leq \epsilon$ , then  $|\hat{x} - x^*|$  may be as large as  $\sqrt{2\epsilon/|f''(x^*)|}$

- Thus, based on function values alone, minima can be computed to only about half precision



## Golden Section Search

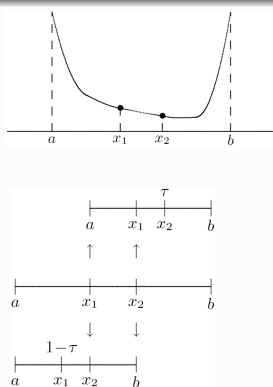
- Suppose  $f$  is unimodal on  $[a, b]$ , and let  $x_1$  and  $x_2$  be two points within  $[a, b]$ , with  $x_1 < x_2$
- Evaluating and comparing  $f(x_1)$  and  $f(x_2)$ , we can discard either  $(x_2, b]$  or  $[a, x_1)$ , with minimum known to lie in remaining subinterval
- To repeat process, we need compute only one new function evaluation
- To reduce length of interval by fixed fraction at each iteration, each new pair of points must have same relationship with respect to new interval that previous pair had with respect to previous interval



## Golden Section Search, continued

```

 $\tau = (\sqrt{5} - 1)/2$ 
 $x_1 = a + (1 - \tau)(b - a); f_1 = f(x_1)$ 
 $x_2 = a + \tau(b - a); f_2 = f(x_2)$ 
while  $((b - a) > tol)$  do
  if  $(f_1 > f_2)$  then
     $a = x_1$ 
     $x_1 = x_2$ 
     $f_1 = f_2$ 
     $x_2 = a + \tau(b - a)$ 
     $f_2 = f(x_2)$ 
  else
     $b = x_2$ 
     $x_2 = x_1$ 
     $f_2 = f_1$ 
     $x_1 = a + (1 - \tau)(b - a)$ 
     $f_1 = f(x_1)$ 
  end
end
    
```



## Example, continued

$x_1$	$f_1$	$x_2$	$f_2$
0.764	0.074	1.236	0.232
0.472	0.122	0.764	0.074
0.764	0.074	0.944	0.113
0.652	0.074	0.764	0.074
0.584	0.085	0.652	0.074
0.652	0.074	0.695	0.071
0.695	0.071	0.721	0.071
0.679	0.072	0.695	0.071
0.695	0.071	0.705	0.071
0.705	0.071	0.711	0.071

< interactive example >



## Unimodality

- For minimizing function of one variable, we need "bracket" for solution analogous to sign change for nonlinear equation
- Real-valued function  $f$  is **unimodal** on interval  $[a, b]$  if there is unique  $x^* \in [a, b]$  such that  $f(x^*)$  is minimum of  $f$  on  $[a, b]$ , and  $f$  is strictly decreasing for  $x \leq x^*$ , strictly increasing for  $x \geq x^*$
- Unimodality enables discarding portions of interval based on sample function values, analogous to interval bisection



## Golden Section Search, continued

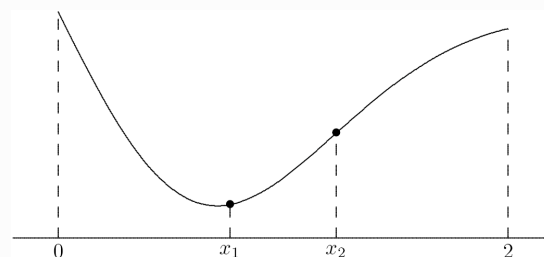
- To accomplish this, we choose relative positions of two points as  $\tau$  and  $1 - \tau$ , where  $\tau^2 = 1 - \tau$ , so  $\tau = (\sqrt{5} - 1)/2 \approx 0.618$  and  $1 - \tau \approx 0.382$
- Whichever subinterval is retained, its length will be  $\tau$  relative to previous interval, and interior point retained will be at position either  $\tau$  or  $1 - \tau$  relative to new interval
- To continue iteration, we need to compute only one new function value, at complementary point
- This choice of sample points is called **golden section search**
- Golden section search is safe but convergence rate is only linear, with constant  $C \approx 0.618$



## Example: Golden Section Search

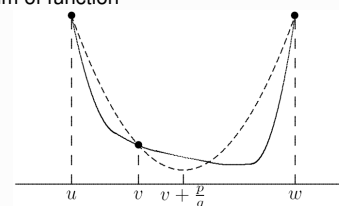
Use golden section search to minimize

$$f(x) = 0.5 - x \exp(-x^2)$$



## Successive Parabolic Interpolation

- Fit quadratic polynomial to three function values
- Take minimum of quadratic to be new approximation to minimum of function



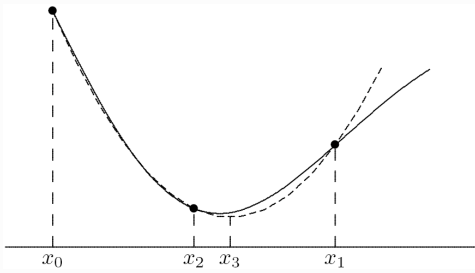
- New point replaces oldest of three previous points and process is repeated until convergence
- Convergence rate of successive parabolic interpolation is superlinear, with  $r \approx 1.324$



## Example: Successive Parabolic Interpolation

Use successive parabolic interpolation to minimize

$$f(x) = 0.5 - x \exp(-x^2)$$



## Newton's Method

- Another local quadratic approximation is truncated Taylor series

$$f(x+h) \approx f(x) + f'(x)h + \frac{f''(x)}{2}h^2$$

- By differentiation, minimum of this quadratic function of  $h$  is given by  $h = -f'(x)/f''(x)$
- Suggests iteration scheme

$$x_{k+1} = x_k - f'(x_k)/f''(x_k)$$

which is **Newton's method** for solving nonlinear equation  $f'(x) = 0$

- Newton's method for finding minimum normally has quadratic convergence rate, but must be started close enough to solution to converge [< interactive example >](#)

## Safeguarded Methods

- As with nonlinear equations in one dimension, slow-but-sure and fast-but-risky optimization methods can be combined to provide both safety and efficiency
- Most library routines for one-dimensional optimization are based on this hybrid approach
- Popular combination is golden section search and successive parabolic interpolation, for which no derivatives are required

## Steepest Descent Method

- Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be real-valued function of  $n$  real variables
- At any point  $x$  where gradient vector is nonzero, negative gradient,  $-\nabla f(x)$ , points downhill toward lower values of  $f$
- In fact,  $-\nabla f(x)$  is locally direction of steepest descent:  $f$  decreases more rapidly along direction of negative gradient than along any other
- Steepest descent** method: starting from initial guess  $x_0$ , successive approximate solutions given by

$$x_{k+1} = x_k - \alpha_k \nabla f(x_k)$$

where  $\alpha_k$  is **line search** parameter that determines how far to go in given direction

## Example, continued

$x_k$	$f(x_k)$
0.000	0.500
0.600	0.081
1.200	0.216
0.754	0.073
0.721	0.071
0.692	0.071
0.707	0.071

[< interactive example >](#)

## Example: Newton's Method

- Use Newton's method to minimize  $f(x) = 0.5 - x \exp(-x^2)$
- First and second derivatives of  $f$  are given by

$$f'(x) = (2x^2 - 1) \exp(-x^2)$$

and

$$f''(x) = 2x(3 - 2x^2) \exp(-x^2)$$

- Newton iteration for zero of  $f'$  is given by

$$x_{k+1} = x_k - (2x_k^2 - 1)/(2x_k(3 - 2x_k^2))$$

- Using starting guess  $x_0 = 1$ , we obtain

$x_k$	$f(x_k)$
1.000	0.132
0.500	0.111
0.700	0.071
0.707	0.071

## Direct Search Methods

- Direct search methods for multidimensional optimization make no use of function values other than comparing them
- For minimizing function  $f$  of  $n$  variables, **Nelder-Mead** method begins with  $n + 1$  starting points, forming **simplex** in  $\mathbb{R}^n$
- Then move to new point along straight line from current point having highest function value through centroid of other points
- New point replaces worst point, and process is repeated
- Direct search methods are useful for nonsmooth functions or for small  $n$ , but expensive for larger  $n$

[< interactive example >](#)

## Steepest Descent, continued

- Given descent direction, such as negative gradient, determining appropriate value for  $\alpha_k$  at each iteration is one-dimensional minimization problem

$$\min_{\alpha_k} f(x_k - \alpha_k \nabla f(x_k))$$

that can be solved by methods already discussed

- Steepest descent method is very reliable: it can always make progress provided gradient is nonzero
- But method is myopic in its view of function's behavior, and resulting iterates can zigzag back and forth, making very slow progress toward solution
- In general, convergence rate of steepest descent is only linear, with constant factor that can be arbitrarily close to 1

## Example: Steepest Descent

- Use steepest descent method to minimize

$$f(\mathbf{x}) = 0.5x_1^2 + 2.5x_2^2$$

- Gradient is given by  $\nabla f(\mathbf{x}) = \begin{bmatrix} x_1 \\ 5x_2 \end{bmatrix}$
- Taking  $\mathbf{x}_0 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ , we have  $\nabla f(\mathbf{x}_0) = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$
- Performing line search along negative gradient direction,

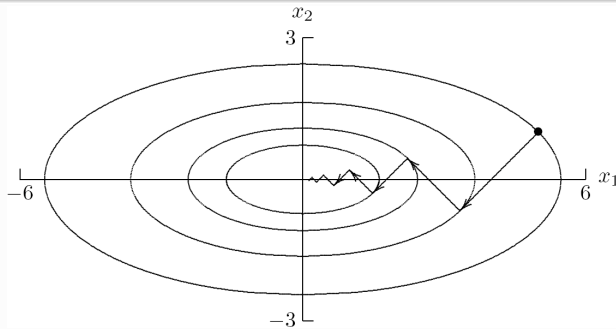
$$\min_{\alpha_0} f(\mathbf{x}_0 - \alpha_0 \nabla f(\mathbf{x}_0))$$

exact minimum along line is given by  $\alpha_0 = 1/3$ , so next

$$\text{approximation is } \mathbf{x}_1 = \begin{bmatrix} 3.333 \\ -0.667 \end{bmatrix}$$



## Example, continued



< interactive example >



## Newton's Method, continued

- Do not explicitly invert Hessian matrix, but instead solve linear system

$$\mathbf{H}_f(\mathbf{x}_k) \mathbf{s}_k = -\nabla f(\mathbf{x}_k)$$

for Newton step  $\mathbf{s}_k$ , then take as next iterate

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{s}_k$$

- Convergence rate of Newton's method for minimization is normally quadratic
- As usual, Newton's method is unreliable unless started close enough to solution to converge

< interactive example >



## Newton's Method, continued

- In principle, line search parameter is unnecessary with Newton's method, since quadratic model determines length, as well as direction, of step to next approximate solution
- When started far from solution, however, it may still be advisable to perform line search along direction of Newton step  $\mathbf{s}_k$  to make method more robust (*damped* Newton)
- Once iterates are near solution, then  $\alpha_k = 1$  should suffice for subsequent iterations



## Example, continued

$\mathbf{x}_k$	$f(\mathbf{x}_k)$	$\nabla f(\mathbf{x}_k)$
5.000 1.000	15.000	5.000 5.000
3.333 -0.667	6.667	3.333 -3.333
2.222 0.444	2.963	2.222 2.222
1.481 -0.296	1.317	1.481 -1.481
0.988 0.198	0.585	0.988 0.988
0.658 -0.132	0.260	0.658 -0.658
0.439 0.088	0.116	0.439 0.439
0.293 -0.059	0.051	0.293 -0.293
0.195 0.039	0.023	0.195 0.195
0.130 -0.026	0.010	0.130 -0.130



## Newton's Method

- Broader view can be obtained by local quadratic approximation, which is equivalent to Newton's method
- In multidimensional optimization, we seek zero of gradient, so *Newton iteration* has form

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \mathbf{H}_f^{-1}(\mathbf{x}_k) \nabla f(\mathbf{x}_k)$$

where  $\mathbf{H}_f(\mathbf{x})$  is Hessian matrix of second partial derivatives of  $f$ ,

$$\{\mathbf{H}_f(\mathbf{x})\}_{ij} = \frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j}$$



## Example: Newton's Method

- Use Newton's method to minimize

$$f(\mathbf{x}) = 0.5x_1^2 + 2.5x_2^2$$

- Gradient and Hessian are given by

$$\nabla f(\mathbf{x}) = \begin{bmatrix} x_1 \\ 5x_2 \end{bmatrix} \quad \text{and} \quad \mathbf{H}_f(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

- Taking  $\mathbf{x}_0 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ , we have  $\nabla f(\mathbf{x}_0) = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$
- Linear system for Newton step is  $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \mathbf{s}_0 = \begin{bmatrix} -5 \\ -5 \end{bmatrix}$ , so  $\mathbf{x}_1 = \mathbf{x}_0 + \mathbf{s}_0 = \begin{bmatrix} 5 \\ 1 \end{bmatrix} + \begin{bmatrix} -5 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ , which is exact solution for this problem, as expected for quadratic function



## Newton's Method, continued

- If objective function  $f$  has continuous second partial derivatives, then Hessian matrix  $\mathbf{H}_f$  is symmetric, and near minimum it is positive definite
- Thus, linear system for step to next iterate can be solved in only about half of work required for LU factorization
- Far from minimum,  $\mathbf{H}_f(\mathbf{x}_k)$  may not be positive definite, so Newton step  $\mathbf{s}_k$  may not be *descent direction* for function, i.e., we may not have

$$\nabla f(\mathbf{x}_k)^T \mathbf{s}_k < 0$$

- In this case, alternative descent direction can be computed, such as negative gradient or direction of negative curvature, and then perform line search



## Trust Region Methods

- Alternative to line search is *trust region method*, in which approximate solution is constrained to lie within region where quadratic model is sufficiently accurate
- If current trust radius is binding, minimizing quadratic model function subject to this constraint may modify direction as well as length of Newton step
- Accuracy of quadratic model is assessed by comparing actual decrease in objective function with that predicted by quadratic model, and trust radius is increased or decreased accordingly



## Quasi-Newton Methods

- Newton's method costs  $\mathcal{O}(n^3)$  arithmetic and  $\mathcal{O}(n^2)$  scalar function evaluations per iteration for dense problem
- Many variants of Newton's method improve reliability and reduce overhead
- Quasi-Newton* methods have form
 
$$\mathbf{x}_{k+1} = \mathbf{x}_k - \alpha_k \mathbf{B}_k^{-1} \nabla f(\mathbf{x}_k)$$
 where  $\alpha_k$  is line search parameter and  $\mathbf{B}_k$  is approximation to Hessian matrix
- Many quasi-Newton methods are more robust than Newton's method, are superlinearly convergent, and have lower overhead per iteration, which often more than offsets their slower convergence rate



## BFGS Method

One of most effective secant updating methods for minimization is *BFGS*

```

x0 = initial guess
B0 = initial Hessian approximation
for k = 0, 1, 2, ...
    Solve Bk sk = -∇f(xk) for sk
    xk+1 = xk + sk
    yk = ∇f(xk+1) - ∇f(xk)
    Bk+1 = Bk + (yk yk^T) / (yk^T sk) - (Bk sk s_k^T Bk) / (s_k^T Bk s_k)
end
    
```



## Example: BFGS Method

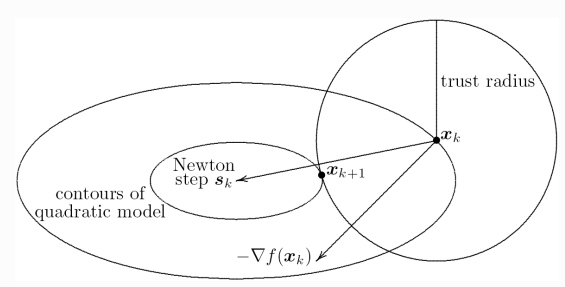
- Use BFGS to minimize  $f(\mathbf{x}) = 0.5x_1^2 + 2.5x_2^2$
- Gradient is given by  $\nabla f(\mathbf{x}) = \begin{bmatrix} x_1 \\ 5x_2 \end{bmatrix}$
- Taking  $\mathbf{x}_0 = [5 \ 1]^T$  and  $\mathbf{B}_0 = \mathbf{I}$ , initial step is negative gradient, so

$$\mathbf{x}_1 = \mathbf{x}_0 + \mathbf{s}_0 = \begin{bmatrix} 5 \\ 1 \end{bmatrix} + \begin{bmatrix} -5 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

- Updating approximate Hessian using BFGS formula, we obtain
 
$$\mathbf{B}_1 = \begin{bmatrix} 0.667 & 0.333 \\ 0.333 & 0.667 \end{bmatrix}$$
- Then new step is computed and process is repeated



## Trust Region Methods, continued



## Secant Updating Methods

- Could use Broyden's method to seek zero of gradient, but this would not preserve symmetry of Hessian matrix
- Several secant updating formulas have been developed for minimization that not only preserve symmetry in approximate Hessian matrix, but also preserve positive definiteness
- Symmetry reduces amount of work required by about half, while positive definiteness guarantees that quasi-Newton step will be descent direction



## BFGS Method, continued

- In practice, factorization of  $\mathbf{B}_k$  is updated rather than  $\mathbf{B}_k$  itself, so linear system for  $\mathbf{s}_k$  can be solved at cost of  $\mathcal{O}(n^2)$  rather than  $\mathcal{O}(n^3)$  work
- Unlike Newton's method for minimization, no second derivatives are required
- Can start with  $\mathbf{B}_0 = \mathbf{I}$ , so initial step is along negative gradient, and then second derivative information is gradually built up in approximate Hessian matrix over successive iterations
- BFGS normally has superlinear convergence rate, even though approximate Hessian does not necessarily converge to true Hessian
- Line search can be used to enhance effectiveness



## Example: BFGS Method

$\mathbf{x}_k$		$f(\mathbf{x}_k)$	$\nabla f(\mathbf{x}_k)$	
5.000	1.000	15.000	5.000	5.000
0.000	-4.000	40.000	0.000	-20.000
-2.222	0.444	2.963	-2.222	2.222
0.816	0.082	0.350	0.816	0.408
-0.009	-0.015	0.001	-0.009	-0.077
-0.001	0.001	0.000	-0.001	0.005

- Increase in function value can be avoided by using line search, which generally enhances convergence
- For quadratic objective function, BFGS with exact line search finds exact solution in at most  $n$  iterations, where  $n$  is dimension of problem **< interactive example >**



## Conjugate Gradient Method

- Another method that does not require explicit second derivatives, and does not even store approximation to Hessian matrix, is **conjugate gradient** (CG) method
- CG generates sequence of conjugate search directions, implicitly accumulating information about Hessian matrix
- For quadratic objective function, CG is theoretically exact after at most  $n$  iterations, where  $n$  is dimension of problem
- CG is effective for general unconstrained minimization as well



## Example: Conjugate Gradient Method

- Use CG method to minimize  $f(\mathbf{x}) = 0.5x_1^2 + 2.5x_2^2$
- Gradient is given by  $\nabla f(\mathbf{x}) = \begin{bmatrix} x_1 \\ 5x_2 \end{bmatrix}$
- Taking  $\mathbf{x}_0 = [5 \ 1]^T$ , initial search direction is negative gradient,
 
$$\mathbf{s}_0 = -\mathbf{g}_0 = -\nabla f(\mathbf{x}_0) = \begin{bmatrix} -5 \\ -5 \end{bmatrix}$$
- Exact minimum along line is given by  $\alpha_0 = 1/3$ , so next approximation is  $\mathbf{x}_1 = [3.333 \ -0.667]^T$ , and we compute new gradient,

$$\mathbf{g}_1 = \nabla f(\mathbf{x}_1) = \begin{bmatrix} 3.333 \\ -3.333 \end{bmatrix}$$



## Truncated Newton Methods

- Another way to reduce work in Newton-like methods is to solve linear system for Newton step by iterative method
- Small number of iterations may suffice to produce step as useful as true Newton step, especially far from overall solution, where true Newton step may be unreliable anyway
- Good choice for linear iterative solver is CG method, which gives step intermediate between steepest descent and Newton-like step
- Since only matrix-vector products are required, explicit formation of Hessian matrix can be avoided by using finite difference of gradient along given vector



## Nonlinear Least Squares, continued

- Linear system for Newton step is

$$\left( \mathbf{J}^T(\mathbf{x}_k)\mathbf{J}(\mathbf{x}_k) + \sum_{i=1}^m r_i(\mathbf{x}_k)\mathbf{H}_i(\mathbf{x}_k) \right) \mathbf{s}_k = -\mathbf{J}^T(\mathbf{x}_k)\mathbf{r}(\mathbf{x}_k)$$

- $m$  Hessian matrices  $\mathbf{H}_i$  are usually inconvenient and expensive to compute
- Moreover, in  $\mathbf{H}_\phi$  each  $\mathbf{H}_i$  is multiplied by residual component  $r_i$ , which is small at solution if fit of model function to data is good



## Conjugate Gradient Method, continued

```

 $\mathbf{x}_0 =$  initial guess
 $\mathbf{g}_0 = \nabla f(\mathbf{x}_0)$ 
 $\mathbf{s}_0 = -\mathbf{g}_0$ 
for  $k = 0, 1, 2, \dots$ 
    Choose  $\alpha_k$  to minimize  $f(\mathbf{x}_k + \alpha_k \mathbf{s}_k)$ 
     $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{s}_k$ 
     $\mathbf{g}_{k+1} = \nabla f(\mathbf{x}_{k+1})$ 
     $\beta_{k+1} = (\mathbf{g}_{k+1}^T \mathbf{g}_{k+1}) / (\mathbf{g}_k^T \mathbf{g}_k)$ 
     $\mathbf{s}_{k+1} = -\mathbf{g}_{k+1} + \beta_{k+1} \mathbf{s}_k$ 
end
    
```

- Alternative formula for  $\beta_{k+1}$  is

$$\beta_{k+1} = ((\mathbf{g}_{k+1} - \mathbf{g}_k)^T \mathbf{g}_{k+1}) / (\mathbf{g}_k^T \mathbf{g}_k)$$



## Example, continued

- So far there is no difference from steepest descent method
- At this point, however, rather than search along new negative gradient, we compute instead

$$\beta_1 = (\mathbf{g}_1^T \mathbf{g}_1) / (\mathbf{g}_0^T \mathbf{g}_0) = 0.444$$

which gives as next search direction

$$\mathbf{s}_1 = -\mathbf{g}_1 + \beta_1 \mathbf{s}_0 = \begin{bmatrix} -3.333 \\ 3.333 \end{bmatrix} + 0.444 \begin{bmatrix} -5 \\ -5 \end{bmatrix} = \begin{bmatrix} -5.556 \\ 1.111 \end{bmatrix}$$

- Minimum along this direction is given by  $\alpha_1 = 0.6$ , which gives exact solution at origin, as expected for quadratic function

< interactive example >



## Nonlinear Least Squares

- Given data  $(t_i, y_i)$ , find vector  $\mathbf{x}$  of parameters that gives "best fit" in least squares sense to model function  $f(t, \mathbf{x})$ , where  $f$  is nonlinear function of  $\mathbf{x}$

- Define components of **residual function**

$$r_i(\mathbf{x}) = y_i - f(t_i, \mathbf{x}), \quad i = 1, \dots, m$$

so we want to minimize  $\phi(\mathbf{x}) = \frac{1}{2} \mathbf{r}^T(\mathbf{x})\mathbf{r}(\mathbf{x})$

- Gradient vector is  $\nabla \phi(\mathbf{x}) = \mathbf{J}^T(\mathbf{x})\mathbf{r}(\mathbf{x})$  and Hessian matrix is

$$\mathbf{H}_\phi(\mathbf{x}) = \mathbf{J}^T(\mathbf{x})\mathbf{J}(\mathbf{x}) + \sum_{i=1}^m r_i(\mathbf{x})\mathbf{H}_i(\mathbf{x})$$

where  $\mathbf{J}(\mathbf{x})$  is Jacobian of  $\mathbf{r}(\mathbf{x})$ , and  $\mathbf{H}_i(\mathbf{x})$  is Hessian of  $r_i(\mathbf{x})$



## Gauss-Newton Method

- This motivates Gauss-Newton method for nonlinear least squares, in which second-order term is dropped and linear system

$$\mathbf{J}^T(\mathbf{x}_k)\mathbf{J}(\mathbf{x}_k)\mathbf{s}_k = -\mathbf{J}^T(\mathbf{x}_k)\mathbf{r}(\mathbf{x}_k)$$

is solved for approximate Newton step  $\mathbf{s}_k$  at each iteration

- This is system of normal equations for linear least squares problem

$$\mathbf{J}(\mathbf{x}_k)\mathbf{s}_k \cong -\mathbf{r}(\mathbf{x}_k)$$

which can be solved better by QR factorization

- Next approximate solution is then given by

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{s}_k$$

and process is repeated until convergence



## Example: Gauss-Newton Method

- Use Gauss-Newton method to fit nonlinear model function

$$f(t, \mathbf{x}) = x_1 \exp(x_2 t)$$

to data

$$\begin{array}{c|cccc} t & 0.0 & 1.0 & 2.0 & 3.0 \\ y & 2.0 & 0.7 & 0.3 & 0.1 \end{array}$$

- For this model function, entries of Jacobian matrix of residual function  $r$  are given by

$$\{\mathbf{J}(\mathbf{x})\}_{i,1} = \frac{\partial r_i(\mathbf{x})}{\partial x_1} = -\exp(x_2 t_i)$$

$$\{\mathbf{J}(\mathbf{x})\}_{i,2} = \frac{\partial r_i(\mathbf{x})}{\partial x_2} = -x_1 t_i \exp(x_2 t_i)$$



## Example, continued

$x_k$		$\ r(x_k)\ _2^2$
1.000	0.000	2.390
1.690	-0.610	0.212
1.975	-0.930	0.007
1.994	-1.004	0.002
1.995	-1.009	0.002
1.995	-1.010	0.002

< interactive example >



## Levenberg-Marquardt Method

- Levenberg-Marquardt method is another useful alternative when Gauss-Newton approximation is inadequate or yields rank deficient linear least squares subproblem

- In this method, linear system at each iteration is of form

$$(\mathbf{J}^T(\mathbf{x}_k)\mathbf{J}(\mathbf{x}_k) + \mu_k \mathbf{I})\mathbf{s}_k = -\mathbf{J}^T(\mathbf{x}_k)\mathbf{r}(\mathbf{x}_k)$$

where  $\mu_k$  is scalar parameter chosen by some strategy

- Corresponding linear least squares problem is

$$\begin{bmatrix} \mathbf{J}(\mathbf{x}_k) \\ \sqrt{\mu_k} \mathbf{I} \end{bmatrix} \mathbf{s}_k \cong \begin{bmatrix} -\mathbf{r}(\mathbf{x}_k) \\ \mathbf{0} \end{bmatrix}$$

- With suitable strategy for choosing  $\mu_k$ , this method can be very robust in practice, and it forms basis for several effective software packages < interactive example >



## Sequential Quadratic Programming

- Foregoing block  $2 \times 2$  linear system is equivalent to quadratic programming problem, so this approach is known as *sequential quadratic programming*

- Types of solution methods include

- Direct solution* methods, in which entire block  $2 \times 2$  system is solved directly
- Range space* methods, based on block elimination in block  $2 \times 2$  linear system
- Null space* methods, based on orthogonal factorization of matrix of constraint normals,  $\mathbf{J}_g^T(\mathbf{x})$

< interactive example >



## Example, continued

- If we take  $\mathbf{x}_0 = [1 \ 0]^T$ , then Gauss-Newton step  $\mathbf{s}_0$  is given by linear least squares problem

$$\begin{bmatrix} -1 & 0 \\ -1 & -1 \\ -1 & -2 \\ -1 & -3 \end{bmatrix} \mathbf{s}_0 \cong \begin{bmatrix} -1 \\ 0.3 \\ 0.7 \\ 0.9 \end{bmatrix}$$

$$\text{whose solution is } \mathbf{s}_0 = \begin{bmatrix} 0.69 \\ -0.61 \end{bmatrix}$$

- Then next approximate solution is given by  $\mathbf{x}_1 = \mathbf{x}_0 + \mathbf{s}_0$ , and process is repeated until convergence



## Gauss-Newton Method, continued

- Gauss-Newton method replaces nonlinear least squares problem by sequence of linear least squares problems whose solutions converge to solution of original nonlinear problem
- If residual at solution is large, then second-order term omitted from Hessian is not negligible, and Gauss-Newton method may converge slowly or fail to converge
- In such "large-residual" cases, it may be best to use general nonlinear minimization method that takes into account true full Hessian matrix



## Equality-Constrained Optimization

- For equality-constrained minimization problem

$$\min f(\mathbf{x}) \quad \text{subject to} \quad \mathbf{g}(\mathbf{x}) = \mathbf{0}$$

where  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  and  $\mathbf{g}: \mathbb{R}^n \rightarrow \mathbb{R}^m$ , with  $m \leq n$ , we seek critical point of Lagrangian  $\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \boldsymbol{\lambda}^T \mathbf{g}(\mathbf{x})$

- Applying Newton's method to nonlinear system

$$\nabla \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = \begin{bmatrix} \nabla f(\mathbf{x}) + \mathbf{J}_g^T(\mathbf{x})\boldsymbol{\lambda} \\ \mathbf{g}(\mathbf{x}) \end{bmatrix} = \mathbf{0}$$

we obtain linear system

$$\begin{bmatrix} \mathbf{B}(\mathbf{x}, \boldsymbol{\lambda}) & \mathbf{J}_g^T(\mathbf{x}) \\ \mathbf{J}_g(\mathbf{x}) & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{s} \\ \boldsymbol{\delta} \end{bmatrix} = - \begin{bmatrix} \nabla f(\mathbf{x}) + \mathbf{J}_g^T(\mathbf{x})\boldsymbol{\lambda} \\ \mathbf{g}(\mathbf{x}) \end{bmatrix}$$

for Newton step  $(\mathbf{s}, \boldsymbol{\delta})$  in  $(\mathbf{x}, \boldsymbol{\lambda})$  at each iteration



## Merit Function

- Once Newton step  $(\mathbf{s}, \boldsymbol{\delta})$  determined, we need *merit function* to measure progress toward overall solution for use in line search or trust region
- Popular choices include *penalty function*

$$\phi_\rho(\mathbf{x}) = f(\mathbf{x}) + \frac{1}{2} \rho \mathbf{g}(\mathbf{x})^T \mathbf{g}(\mathbf{x})$$

and *augmented Lagrangian function*

$$\mathcal{L}_\rho(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \boldsymbol{\lambda}^T \mathbf{g}(\mathbf{x}) + \frac{1}{2} \rho \mathbf{g}(\mathbf{x})^T \mathbf{g}(\mathbf{x})$$

where parameter  $\rho > 0$  determines relative weighting of optimality vs feasibility

- Given starting guess  $\mathbf{x}_0$ , good starting guess for  $\boldsymbol{\lambda}_0$  can be obtained from least squares problem

$$\mathbf{J}_g^T(\mathbf{x}_0) \boldsymbol{\lambda}_0 \cong -\nabla f(\mathbf{x}_0)$$



## Inequality-Constrained Optimization

- Methods just outlined for equality constraints can be extended to handle inequality constraints by using *active set* strategy
- Inequality constraints are provisionally divided into those that are satisfied already (and can therefore be temporarily disregarded) and those that are violated (and are therefore temporarily treated as equality constraints)
- This division of constraints is revised as iterations proceed until eventually correct constraints are identified that are binding at solution



## Barrier Methods

- For inequality-constrained problems, another alternative is *barrier function*, such as

$$\phi_\mu(\mathbf{x}) = f(\mathbf{x}) - \mu \sum_{i=1}^p \frac{1}{h_i(\mathbf{x})}$$

or

$$\phi_\mu(\mathbf{x}) = f(\mathbf{x}) - \mu \sum_{i=1}^p \log(-h_i(\mathbf{x}))$$

which increasingly penalize feasible points as they approach boundary of feasible region

- Again, solutions of unconstrained problem approach  $\mathbf{x}^*$  as  $\mu \rightarrow 0$ , but problems are increasingly ill-conditioned, so solve sequence of problems with decreasing values of  $\mu$
- Barrier functions are basis for *interior point* methods for linear programming



## Example, continued

- So system to be solved for critical point of Lagrangian is

$$\begin{aligned} x_1 + \lambda &= 0 \\ 5x_2 - \lambda &= 0 \\ x_1 - x_2 &= 1 \end{aligned}$$

which in this case is linear system

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 5 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Solving this system, we obtain solution

$$x_1 = 0.833, \quad x_2 = -0.167, \quad \lambda = -0.833$$



## Linear Programming

- One of most important and common constrained optimization problems is *linear programming*
- One standard form for such problems is

$$\min f(\mathbf{x}) = \mathbf{c}^T \mathbf{x} \quad \text{subject to} \quad \mathbf{Ax} = \mathbf{b} \quad \text{and} \quad \mathbf{x} \geq \mathbf{0}$$

where  $m < n$ ,  $\mathbf{A} \in \mathbb{R}^{m \times n}$ ,  $\mathbf{b} \in \mathbb{R}^m$ , and  $\mathbf{c}, \mathbf{x} \in \mathbb{R}^n$

- Feasible region is convex polyhedron in  $\mathbb{R}^n$ , and minimum must occur at one of its vertices
- *Simplex* method moves systematically from vertex to vertex until minimum point is found



## Penalty Methods

- Merit function can also be used to convert equality-constrained problem into sequence of unconstrained problems
- If  $\mathbf{x}_\rho^*$  is solution to

$$\min_{\mathbf{x}} \phi_\rho(\mathbf{x}) = f(\mathbf{x}) + \frac{1}{2} \rho \mathbf{g}(\mathbf{x})^T \mathbf{g}(\mathbf{x})$$

then under appropriate conditions

$$\lim_{\rho \rightarrow \infty} \mathbf{x}_\rho^* = \mathbf{x}^*$$

- This enables use of unconstrained optimization methods, but problem becomes ill-conditioned for large  $\rho$ , so we solve sequence of problems with gradually increasing values of  $\rho$ , with minimum for each problem used as starting point for next problem [< interactive example >](#)



## Example: Constrained Optimization

- Consider quadratic programming problem

$$\min_{\mathbf{x}} f(\mathbf{x}) = 0.5x_1^2 + 2.5x_2^2$$

subject to

$$\mathbf{g}(\mathbf{x}) = x_1 - x_2 - 1 = 0$$

- Lagrangian function is given by

$$\mathcal{L}(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda \mathbf{g}(\mathbf{x}) = 0.5x_1^2 + 2.5x_2^2 + \lambda(x_1 - x_2 - 1)$$

- Since

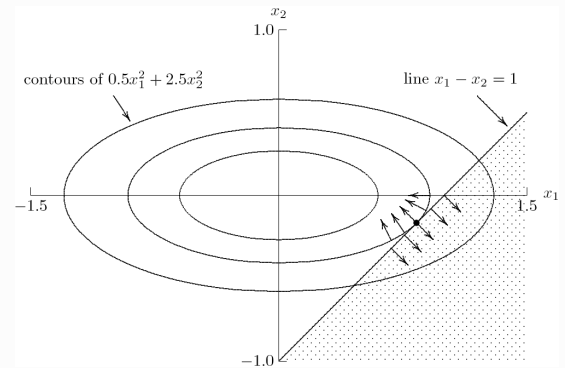
$$\nabla f(\mathbf{x}) = \begin{bmatrix} x_1 \\ 5x_2 \end{bmatrix} \quad \text{and} \quad \mathbf{J}_g(\mathbf{x}) = [1 \quad -1]$$

we have

$$\nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \lambda) = \nabla f(\mathbf{x}) + \mathbf{J}_g^T(\mathbf{x}) \lambda = \begin{bmatrix} x_1 \\ 5x_2 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



## Example, continued



## Linear Programming, continued

- Simplex method is reliable and normally efficient, able to solve problems with thousands of variables, but can require time exponential in size of problem in worst case
- *Interior point* methods for linear programming developed in recent years have polynomial worst case solution time
- These methods move through interior of feasible region, not restricting themselves to investigating only its vertices
- Although interior point methods have significant practical impact, simplex method is still predominant method in standard packages for linear programming, and its effectiveness in practice is excellent



## Example: Linear Programming

- To illustrate linear programming, consider

$$\min_{\mathbf{x}} \mathbf{c}^T \mathbf{x} = -8x_1 - 11x_2$$

subject to linear inequality constraints

$$5x_1 + 4x_2 \leq 40, \quad -x_1 + 3x_2 \leq 12, \quad x_1 \geq 0, \quad x_2 \geq 0$$

- Minimum value must occur at vertex of feasible region, in this case at  $x_1 = 3.79$ ,  $x_2 = 5.26$ , where objective function has value  $-88.2$



## Example, continued

