

Application and Discussion of The LDG And Other DG Methods

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1 Introduction

This paper is meant to make it clear how to implement the DG methods for Elliptic problems discussed in [1]. The LDG method will be discussed in detail. With a clear understanding of the LDG method, correct implementation of the other DG methods should only require an understanding of how to implement their fluxes.

2 LDG Applied to the Helmholtz Equation

In this section, we derive the weak and the strong form of the Helmholtz equation for the LDG method. We would like to apply the local discontinuous Galerkin (LDG) method to

$$\begin{aligned} \nabla^2 u + \omega^2 u &= f && \text{in } \Omega, \\ u &= g_D && \text{on } \Gamma_D, \\ \nabla u \cdot \mathbf{n} &= \mathbf{g}_N \cdot \mathbf{n} && \text{on } \Gamma_N, \end{aligned} \tag{1}$$

where Ω is a bounded domain in \mathbb{R}^d , \mathbf{n} is the unit outward normal to the boundary, $\Gamma = \Gamma_D \cup \Gamma_N$, f is a given function in $L^2(\Omega)$, a and c map from $\mathbb{R}^d \rightarrow \mathbb{R}$ and \mathbf{b} maps from $\mathbb{R}^d \rightarrow \mathbb{R}^d$

Following the derivations given in [1] and [4], we now derive the weak and the strong forms according to the LDG method. We first write the above system as two first order systems. Intuitively, we consider two first order systems, because in this context defining fluxes is more natural, i.e. it is more natural to think of

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fluxes in the context of a hyperbolic-looking problem.

$$\mathbf{q} - \nabla u = 0 \quad \text{in } \Omega, \quad (2)$$

$$\nabla \cdot \mathbf{q} + \omega^2 u = f \quad \text{in } \Omega, \quad (3)$$

$$u = g_{\mathcal{D}} \quad \text{on } \Gamma_{\mathcal{D}},$$

$$\mathbf{q} \cdot \mathbf{n} = \mathbf{g}_{\mathcal{N}} \cdot \mathbf{n} \quad \text{on } \Gamma_{\mathcal{N}}$$

2.1 “Broken” Spaces

When we triangulate Ω and then discretize with discontinuous elements, the norms and finite-element spaces become “broken.” Given a triangulation \mathcal{T} of Ω , the broken H^1 space is

$$H^1(\Omega, \mathcal{T}) = \{u \in L^2(\Omega) : u|_K \in H^1(K), K \in \mathcal{T}, K \subset \Omega\}, \quad (4)$$

with broken H^1 norm

$$\|u\|_{H^1(\Omega, \mathcal{T})}^2 = \sum_{K \in \mathcal{T}} \|u\|_{H^1(K)}^2. \quad (5)$$

2.2 Weak Formulation

Next in order to obtain the weak formulation, we multiply (2) and (3) by smooth test functions, $\boldsymbol{\psi}$ and v respectively. Then, we integrate by parts over a triangulation, $\mathcal{T} = \{K\}$ of Ω .

$$\int_K \mathbf{q} \cdot \boldsymbol{\psi} \, d\mathbf{x} = - \int_K u \nabla \cdot \boldsymbol{\psi} \, d\mathbf{x} + \int_{\partial K} u \boldsymbol{\psi} \cdot \mathbf{n}_K \, ds \quad (6)$$

$$\int_K -\mathbf{q} \cdot \nabla v + \omega^2 u v \, d\mathbf{x} + \int_{\partial K} v \mathbf{q} \cdot \mathbf{n}_K \, ds = \int_K f v \, d\mathbf{x}, \quad (7)$$

where \mathbf{n}_K is the unit outward normal on K . Now, we introduce the finite element spaces associated with \mathcal{T} . Let

$$\mathbf{V}_h := \{v \in L^2(\Omega) : v|_K \in \mathcal{P}_{\mathcal{P}}(K), \forall K \in \mathcal{T}\} \quad (8)$$

$$\boldsymbol{\Psi}_h := \{\boldsymbol{\psi} \in (L^2(\Omega))^d : \boldsymbol{\psi}|_K \in (\mathcal{P}_{\mathcal{P}}(K))^d, \forall K \in \mathcal{T}\}, \quad (9)$$

where $\mathcal{P}_{\mathcal{P}}(K)$ is the space of all polynomial functions on K of degree at most $\mathcal{P} \geq 1$. Then we replace our exact solution, (u, \mathbf{q}) , with an approximate solution, (u_h, \mathbf{q}_h) . The weak formulation is find $u_h \in \mathbf{V}_h$ and $\mathbf{q}_h \in \boldsymbol{\Psi}_h$ such that $\forall K \in \mathcal{T}$,

$$\int_K \mathbf{q}_h \cdot \boldsymbol{\psi} \, d\mathbf{x} = - \int_K u_h \nabla \cdot \boldsymbol{\psi} \, d\mathbf{x} + \int_{\partial K} \hat{u}_h \boldsymbol{\psi} \cdot \mathbf{n}_K \, ds, \quad (10)$$

$$\int_K -\mathbf{q}_h \cdot \nabla v + \omega^2 u_h v \, d\mathbf{x} + \int_{\partial K} v \hat{\mathbf{q}}_h \cdot \mathbf{n}_K \, ds = \int_K f v \, d\mathbf{x}, \quad (11)$$

holds $\forall v \in \mathbf{V}_h$ and $\forall \boldsymbol{\psi} \in \boldsymbol{\Psi}_h$. We define the numerical fluxes, \hat{u}_h and $\hat{\mathbf{q}}_h$ on the boundary of each $K \in \mathcal{T}$ to be a discrete approximation to the traces of u_h and \mathbf{q}_h respectively on that boundary.

2.3 Numerical Fluxes

Let K^+ and K^- be two adjacent elements of \mathcal{T} , \mathbf{n}^+ and \mathbf{n}^- be the corresponding outward normals and (\mathbf{q}^\pm, u^\pm) be the traces of (\mathbf{q}, u) on the boundary of K^\pm . The fluxes are defined using the jump, $[[\cdot]]$, and the average, $\{\{\cdot\}\}$,

$$\begin{aligned}\{\{u\}\} &:= (u^+ + u^-)/2, & \{\{\mathbf{q}\}\} &:= (\mathbf{q}^+ + \mathbf{q}^-)/2, \\ [[u]] &:= u^+ \mathbf{n}^+ + u^- \mathbf{n}^-, & [[\mathbf{q}]] &:= \mathbf{q}^+ \cdot \mathbf{n}^+ + \mathbf{q}^- \cdot \mathbf{n}^-.\end{aligned}$$

According to [1], the LDG method defines the fluxes as

$$\begin{aligned}\hat{\mathbf{q}} &:= \{\{\mathbf{q}\}\} + \beta [[\mathbf{q}]] - \tau [[u]] & (12) \\ \hat{u} &:= \{\{u\}\} - \beta \cdot [[u]]. & (13)\end{aligned}$$

According to [4], we let $\hat{\mathbf{q}}$ and \hat{u} be the same for adjacent elements over shared edges. This is done by defining β to be the same for any two elements over a shared edge. For all edges, e , such that $e = \partial K^+ \cap \partial K^-$,

$$|\beta(e) \cdot \mathbf{n}^\pm| = \frac{1}{2}. \quad (14)$$

Typically, $\beta(e) = \mathbf{n}^-/2$ or $\mathbf{n}^+/2$. Defining β to be the same for any two elements over a shared edge ensures symmetry of the operator. This is done by keeping the flux consistent, i.e. the flux is the same on the shared edge of adjacent elements.

According to [4] and [3], the fluxes on the boundary, Γ , are defined as

$$\hat{\mathbf{q}} := \begin{cases} \mathbf{q}^- - \tau(u^- - g_{\mathcal{D}})\mathbf{n}^- & \text{on } \Gamma_{\mathcal{D}} \\ \mathbf{g}_{\mathcal{N}} & \text{on } \Gamma_{\mathcal{N}} \end{cases} \quad \hat{u} := \begin{cases} u^+ = g_{\mathcal{D}} & \text{on } \Gamma_{\mathcal{D}} \\ u^- & \text{on } \Gamma_{\mathcal{N}}, \end{cases} \quad (15)$$

where $^+$ denotes the domains exterior and $^-$ denotes the domains interior, i.e. the current element. These values for the fluxes on the boundary can be derived from the general flux definitions, (12) and (13), by allowing $\beta = \mathbf{n}^-/2$ on $\Gamma_{\mathcal{D}}$, $\beta = -\mathbf{n}^-/2$ on $\Gamma_{\mathcal{N}}$, $(q^-, u^+) = (q^+, g_{\mathcal{D}})$ on $\Gamma_{\mathcal{D}}$ and $(q^+, u^+) = (\mathbf{g}_{\mathcal{N}}, u^-)$ on $\Gamma_{\mathcal{N}}$. For more information see also [3] and section 3.

2.4 Solution Technique

2.4.1 Weak Form

A suggested solution technique in 2-D to equations, (10) and (11), is to consider breaking up the basis vector, $\boldsymbol{\psi}$, into two basis vectors, $\boldsymbol{\psi}_x = [v, 0]^T$ and $\boldsymbol{\psi}_y = [0, v]^T$. Then one can write equation (10) as

$$\int_K q_{h,x} v \, d\mathbf{x} = - \int_K u_h \partial_x v \, d\mathbf{x} + \int_{\partial K} \hat{u}_h v n_{K,x} \, ds, \quad (16)$$

$$\int_K q_{h,y} v \, d\mathbf{x} = - \int_K u_h \partial_y v \, d\mathbf{x} + \int_{\partial K} \hat{u}_h v n_{K,y} \, ds, \quad (17)$$

where x and y denote x and y components, not partial derivatives. After solving for $q_{h,x}$ and $q_{h,y}$, one can substitute those values into equation (11) and solve for u .

2.4.2 Strong-Weak Form

To obtain the strong-weak form, we integrate by parts again on equations (10) and (11).

$$\int_K \mathbf{q}_h \cdot \boldsymbol{\psi} \, d\mathbf{x} = \int_K \nabla u_h \cdot \boldsymbol{\psi} \, d\mathbf{x} + \int_{\partial K} (\hat{u}_h - u_h) \mathbf{n}_K \cdot \boldsymbol{\psi} \, ds, \quad (18)$$

$$\int_K \nabla \cdot \mathbf{q}_h v + \omega^2 u_h v \, d\mathbf{x} + \int_{\partial K} v (\hat{\mathbf{q}}_h - \mathbf{q}_h) \cdot \mathbf{n}_K \, ds = \int_K f v \, d\mathbf{x}. \quad (19)$$

This is called the strong-weak form because we have made the differentiability requirements on our solution higher than necessary. Also, note that we do not approximate the traces of u_h or \mathbf{q}_h on elemental boundaries with numerical fluxes when we integrate by parts the second time. Instead, we let u_h and \mathbf{q}_h simply be u_h^- and \mathbf{q}_h^- on elemental boundaries.

Like with the weak form, a solution technique in 2-D to equations, (18) and (19), is to consider breaking up the basis vector, $\boldsymbol{\psi}$, into two basis vectors, $\boldsymbol{\psi}_x = [v, 0]^T$ and $\boldsymbol{\psi}_y = [0, v]^T$. Then one can write equation (18) as

$$\int_K q_{h,x} v \, d\mathbf{x} = \int_K \partial_x u_h v \, d\mathbf{x} + \int_{\partial K} (\hat{u}_h - u_h) v n_{K,x} \, ds, \quad (20)$$

$$\int_K q_{h,y} v \, d\mathbf{x} = \int_K \partial_y u_h v \, d\mathbf{x} + \int_{\partial K} (\hat{u}_h - u_h) v n_{K,y} \, ds, \quad (21)$$

where x and y denote x and y components, not partial derivatives. After solving for $q_{h,x}$ and $q_{h,y}$, one can substitute those values into equation (19) and solve for u . Note that when solving, let u_h and $q_{h,*}$ over an elemental boundary be u_h^- and $q_{h,*}^-$.

3 Boundary Fluxes

In this section, we derive the values for the fluxes on the boundary. Remember that LDG fluxes are defined using the jump, $[[\cdot]]$, and the average, $\{\{\cdot\}\}$, to give us

$$\hat{\mathbf{q}} := \{\{\mathbf{q}\}\} + \beta [[\mathbf{q}]] - \tau [[u]]$$

$$\hat{u} := \{\{u\}\} - \beta \cdot [[u]].$$

According to [3] we let

$$(\mathbf{q}^-, u^+) = (\mathbf{q}^+, g_D) \text{ on } \Gamma_D. \quad (22)$$

$$(\mathbf{q}^+, u^+) = (g_N, u^-) \text{ on } \Gamma_N. \quad (23)$$

On $\Gamma_{\mathcal{D}}$, letting $\mathbf{q}^+ = \mathbf{q}^-$ is like a linear extension of u out into space. Then, we let $u^+ = g_{\mathcal{D}}$ because we linearly extend u^- out to u^+ . On $\Gamma_{\mathcal{N}}$, letting $\mathbf{q}^+ = \mathbf{g}_{\mathcal{N}}$ is an intuitive application of Neumann boundary conditions. However, letting $u^+ = u^-$ is unintuitive, because it implies a constant extension of u out into space so one would think that $\mathbf{q}^- = \mathbf{g}_{\mathcal{N}}$ and $\mathbf{q}^+ = 0$.

By using these assumptions about u^{\pm} and \mathbf{q}^{\pm} on Γ , we can derive appropriate values for β and the fluxes on Γ . We can derive that

$$\hat{\mathbf{q}} := \begin{cases} \mathbf{q}^- - \tau(u^- - g_{\mathcal{D}})\mathbf{n}^- & \text{on } \Gamma_{\mathcal{D}} \\ \mathbf{g}_{\mathcal{N}} & \text{on } \Gamma_{\mathcal{N}} \end{cases} \quad \hat{u} := \begin{cases} u^+ = g_{\mathcal{D}} & \text{on } \Gamma_{\mathcal{D}} \\ u^- & \text{on } \Gamma_{\mathcal{N}} \end{cases}, \quad (24)$$

where $^+$ denotes the domains exterior and $^-$ denotes the domains interior, i.e. the current element. These definitions can be found in section 2.3 and in [4] and [3].

3.1 Dirichlet Boundaries

We now derive \hat{u} and $\hat{\mathbf{q}}$ on $\Gamma_{\mathcal{D}}$. We will see that with $\beta = \mathbf{n}^-/2$ on $\Gamma_{\mathcal{D}}$ the derivation works out correctly. Flipping the sign of β to a negative results in $\hat{u} = u^-$ on $\Gamma_{\mathcal{D}}$.

$$\begin{aligned} \hat{\mathbf{q}} &= \{\{\mathbf{q}\}\} + \beta\llbracket\mathbf{q}\rrbracket - \tau\llbracket u \rrbracket \\ &= \frac{\mathbf{q}^- + \mathbf{q}^+}{2} + \frac{\mathbf{n}^-}{2}(\mathbf{q}^- \cdot \mathbf{n}^-) + \frac{\mathbf{n}^-}{2}(\mathbf{q}^+ \cdot \mathbf{n}^+) - \tau\llbracket u \rrbracket \\ &\quad \text{Note that } \frac{\mathbf{n}^-}{2}(\mathbf{q}^- \cdot \mathbf{n}^-) \neq \left(\frac{\mathbf{n}^-}{2} \cdot \mathbf{n}^-\right) \mathbf{q}^-. \\ &\quad \text{Using } \mathbf{q}^- = \mathbf{q}^+ \text{ on } \Gamma_{\mathcal{D}} \text{ and } \mathbf{n}^- = -\mathbf{n}^+, \\ &= \mathbf{q}^- + \frac{\mathbf{n}^-}{2}(\mathbf{q}^- \cdot \mathbf{n}^-) - \frac{\mathbf{n}^-}{2}(\mathbf{q}^- \cdot \mathbf{n}^-) - \tau\llbracket u \rrbracket \\ &= \mathbf{q}^- - \tau\llbracket u \rrbracket \\ &= \mathbf{q}^- - \tau(u^- \mathbf{n}^- + g_{\mathcal{D}}\mathbf{n}^+) \\ &= \mathbf{q}^- - \tau(u^- \mathbf{n}^- - g_{\mathcal{D}}\mathbf{n}^-) \end{aligned}$$

$$\begin{aligned} \hat{u} &= \{\{u\}\} - \beta \cdot \llbracket u \rrbracket \\ &= \frac{u^+ + u^-}{2} - \frac{\mathbf{n}^-}{2} \cdot (u^+ \mathbf{n}^+ + u^- \mathbf{n}^-) \\ &\quad \text{Using } \mathbf{n}^- = -\mathbf{n}^+, \\ &= \frac{u^+ + u^-}{2} - \frac{1}{2}(u^- - u^+) \\ &= u^+ = g_{\mathcal{D}} \end{aligned}$$

Note that the sign of β does not affect the result of the derivation of $\hat{\mathbf{q}}$, but that the sign of β does determine whether we get $\hat{u} = u^+$ or $\hat{u} = u^-$ on $\Gamma_{\mathcal{D}}$. Also,

it was experimentally observed that the sign of β affected symmetry, with the above formulation being the symmetric one.

It is also interesting to note what the fluxes look like for the strong-weak form along $\Gamma_{\mathcal{D}}$.

$$\hat{u} - u = g_{\mathcal{D}} - u \quad (25)$$

$$\mathbf{n}^- \cdot (\hat{\mathbf{q}} - \mathbf{q}) = \tau(g_{\mathcal{D}} - u^-) \quad (26)$$

We have something that resembles a penalty method. How strong the boundary conditions are imposed though, is an open question.

3.2 Neumann Boundaries

We now do the same type of derivation on $\Gamma_{\mathcal{N}}$. We will see that with $\beta = -n^-/2$ on $\Gamma_{\mathcal{N}}$ the derivation works out correctly. Flipping the sign of β to a positive results in $\hat{\mathbf{q}} = \mathbf{q}^-$ on $\Gamma_{\mathcal{N}}$. We also work with $\hat{\mathbf{q}} \cdot \mathbf{n}^-$, as this is a more natural formulation for $\hat{\mathbf{q}}$ because this is how $\hat{\mathbf{q}}$ appears in the bilinear form.

$$\begin{aligned} \hat{\mathbf{q}} \cdot \mathbf{n}^- &= (\{\mathbf{q}\} + \beta[\mathbf{q}] - \tau[u]) \cdot \mathbf{n}^- \\ &= \mathbf{n}^- \cdot \left(\frac{\mathbf{q}^- + \mathbf{q}^+}{2} \right) - \mathbf{n}^- \cdot \left(\frac{\mathbf{n}^-}{2}(\mathbf{q}^- \cdot \mathbf{n}^-) + \frac{\mathbf{n}^-}{2}(\mathbf{q}^+ \cdot \mathbf{n}^+) \right) - \mathbf{n}^- \cdot \tau[u] \\ &\quad \text{Using } \mathbf{q}^+ = \mathbf{g}_{\mathcal{N}} \text{ on } \Gamma_{\mathcal{N}}, \\ &= \frac{\mathbf{n}^-}{2} \cdot \mathbf{g}_{\mathcal{N}} + \frac{\mathbf{n}^-}{2} \cdot \mathbf{q}^- - \frac{\mathbf{n}^-}{2} \cdot \mathbf{q}^- + \frac{\mathbf{n}^-}{2} \cdot \mathbf{g}_{\mathcal{N}} - \mathbf{n}^- \cdot \tau[u] \\ &= \mathbf{n}^- \cdot \mathbf{g}_{\mathcal{N}} - \mathbf{n}^- \cdot \tau[u] \\ &= \mathbf{n}^- \cdot \mathbf{g}_{\mathcal{N}} - \mathbf{n}^- \cdot (\tau(u^- \mathbf{n}^- + u^+ \mathbf{n}^+)) \\ &\quad \text{Using } u^- = u^+ \text{ on } \Gamma_{\mathcal{N}} \text{ and } \mathbf{n}^- = -\mathbf{n}^+, \\ &= \mathbf{n}^- \cdot \mathbf{g}_{\mathcal{N}} - \mathbf{n}^- \cdot (\tau(u^- \mathbf{n}^- - u^- \mathbf{n}^-)) \\ &= \mathbf{n}^- \cdot \mathbf{g}_{\mathcal{N}} \\ &\Leftrightarrow \\ &\hat{\mathbf{q}} = \mathbf{g}_{\mathcal{N}} \end{aligned}$$

$$\begin{aligned} \hat{u} &= \{u\} - \beta \cdot [u] \\ &= \frac{u^+ + u^-}{2} - \frac{\mathbf{n}^-}{2} \cdot (u^+ \mathbf{n}^+ + u^- \mathbf{n}^-) \\ &\quad \text{Using } u^- = u^+ \text{ on } \Gamma_{\mathcal{N}} \text{ and } \mathbf{n}^- = -\mathbf{n}^+, \\ &= u^- - \frac{\mathbf{n}^-}{2} \cdot (-u^- \mathbf{n}^- + u^- \mathbf{n}^-) \\ &= u^- = u^+ \end{aligned}$$

Note that the sign of β does not affect the result of the derivation of \hat{u} , but that the sign of β does determine our result for $\hat{\mathbf{q}}$.

It is also interesting to note what the fluxes look like for the strong-weak form along $\Gamma_{\mathcal{N}}$.

$$\hat{u} - u = u^- - u^- = 0 \quad (27)$$

$$\mathbf{n}^- \cdot (\hat{\mathbf{q}} - \mathbf{q}) = \mathbf{n}^- \cdot (\mathbf{g}_{\mathcal{N}} - \mathbf{q}^-) \quad (28)$$

We have something that resembles a penalty method for \mathbf{q} .

4 Derivation of Matrix Equations

In this section we do a loose derivation of high-level matrix equations for solving the 2-D Helmholtz problem. We assume all zero Dirichlet boundary conditions.

4.1 Strong-Weak Helmholtz Problem Matrix Equations

We first derive equations for each of the two components of \mathbf{q} . We use equations, (20) and (21) which are again,

$$\begin{aligned} \int_K q_{h,x} v \, d\mathbf{x} &= \int_K \partial_x u_h v \, d\mathbf{x} + \int_{\partial K} (\hat{u}_h - u_h) v n_{K,x} \, ds, \\ \int_K q_{h,y} v \, d\mathbf{x} &= \int_K \partial_y u_h v \, d\mathbf{x} + \int_{\partial K} (\hat{u}_h - u_h) v n_{K,y} \, ds. \end{aligned}$$

In matrix form these equations are respectively,

$$J M \mathbf{q}_x = J M D_x \mathbf{u} + B S J N_x FluxU \mathbf{u} \quad (29)$$

$$J M \mathbf{q}_y = J M D_y \mathbf{u} + B S J N_y FluxU \mathbf{u}, \quad (30)$$

where J is the Jacobian, M is the mass matrix, D_* is a differentiation matrix with respect to x or y , B is a boundary integration matrix, SJ is the surface Jacobian, N_* is a diagonal matrix of the x or y component of the unit outward normal, \mathbf{n}^- . $FluxU \mathbf{u} = [(\hat{u} - u)_i]$, a vector where $(\cdot)_i$ means evaluated at global degree of freedom, i . $B S J \mathbf{u} = [(u, v_i)_{\partial \mathcal{T}}]$ is also a vector, where v_i is the i -th global basis function and $\partial \mathcal{T}$ refers to all elemental and domain boundaries for the triangulation, \mathcal{T} .

Next, we solve for \mathbf{q} .

$$\mathbf{q}_x = D_x \mathbf{u} + (J M)^{-1} B S J N_x FluxU \mathbf{u} = Q_x \mathbf{u} \quad (31)$$

$$\mathbf{q}_y = D_y \mathbf{u} + (J M)^{-1} B S J N_y FluxU \mathbf{u} = Q_y \mathbf{u}. \quad (32)$$

Now that we have \mathbf{q} expressed in terms of \mathbf{u} , we can solve equation, (19), which is again,

$$\int_K \nabla \cdot \mathbf{q}_h v + \omega^2 u v \, d\mathbf{x} + \int_{\partial K} v (\hat{\mathbf{q}}_h - \mathbf{q}_h) \cdot \mathbf{n}_K \, ds = \int_K f v \, d\mathbf{x}$$

We now separate $\hat{\mathbf{q}} - \mathbf{q}$ into its independent parts of u_h and \mathbf{q}_h ; letting $\hat{\mathbf{q}} = \{\{\mathbf{q}\}\} + \beta\llbracket\mathbf{q}\rrbracket$.

$$\int_K \nabla \cdot \mathbf{q}_h v + \omega^2 u v \, d\mathbf{x} + \int_{\partial K} v (\hat{\mathbf{q}}_h - \mathbf{q}_h) \cdot \mathbf{n}_K - v \tau \llbracket u \rrbracket \cdot \mathbf{n}_K \, ds = \int_K f v \, d\mathbf{x}.$$

Finally, we can derive an equation solvable for u .

$$[JM(D_x Q_x + D_y Q_y + \omega^2) + BSJ FluxQ - \tau BSJ JumpU] \mathbf{u} = JMF,$$

where $FluxQ \mathbf{u} = [((\hat{\mathbf{q}} - \mathbf{q}) \cdot \mathbf{n}^-)_i]$, $JumpU \mathbf{u} = [\llbracket u \rrbracket \cdot \mathbf{n}^-]_i$, and $F = [(f)_i]$ are vectors and $(\cdot)_i$ means evaluated at global degree of freedom, i .

A little more discussion of the $FluxU$ matrices is warranted. The $FluxQ$ matrices are analogous. In general, the $Flux^*$ and $Jump^*$ matrices will map a vector of values over the domain interior to a vector of values over element edges. They will be in general rectangular matrices.

For constant order polynomials on conforming meshes, the $FluxU$ matrices will consist of only 0's, $1/2$'s and $-1/2$'s, as we are simply averaging and differencing values across edges where the nodes are at the same geometric location. There are two essential matrices needed for constructing the $FluxU$ matrix,

$$U^+ u = u^+ \tag{33}$$

$$U^- u = u^- \tag{34}$$

By taking linear combinations of these matrices, the flux values at nodal points for u , q_x and q_y can easily be constructed. For instance $\{\{u_h\}\} = 0.5 * (U^+ + U^-)u_h$. As already noted, for the constant order and conforming mesh case, U^+ and U^- are simply various combinations of columns of the Identity. The Multi-P case on a conforming mesh is discussed below in the Multi-P section, but it essentially consists of defining the U^+ matrix using interpolation to map nodes on an opposing edge to the local edge.

The flux matrices will also need to inject u_h and \mathbf{q}_h onto the nodes on the domain boundary in order to handle the boundary flux terms.

4.2 Weak Helmholtz Problem Matrix Equations

We first derive equations for each of the two components of \mathbf{q} . We use equations, (16) and (17) which are again,

$$\begin{aligned} \int_K q_{h,x} v \, d\mathbf{x} &= - \int_K u_h \partial_x v \, d\mathbf{x} + \int_{\partial K} \hat{u}_h v n_{K,x} \, ds, \\ \int_K q_{h,y} v \, d\mathbf{x} &= - \int_K u_h \partial_y v \, d\mathbf{x} + \int_{\partial K} \hat{u}_h v n_{K,y} \, ds, \end{aligned}$$

In matrix form these equations are respectively, while noting appropriate transposes,

$$JM \mathbf{q}_x = -J(M D_x)^T \mathbf{u} + BSJ N_x FluxU \mathbf{u} \tag{35}$$

$$JM \mathbf{q}_y = -J(M D_y)^T \mathbf{u} + BSJ N_y FluxU \mathbf{u}, \tag{36}$$

where matrices are defined as before, except now, $FluxU \mathbf{u} = [(\hat{u})_i]$

Next, we solve for \mathbf{q} .

$$\mathbf{q}_x = -M^{-1}(M D_x)^T \mathbf{u} + (J M)^{-1} B S J N_x FluxU \mathbf{u} = Q_x \mathbf{u} \quad (37)$$

$$\mathbf{q}_y = -M^{-1}(M D_y)^T \mathbf{u} + (J M)^{-1} B S J N_y FluxU \mathbf{u} = Q_y \mathbf{u}. \quad (38)$$

Now that we have \mathbf{q} expressed in terms of \mathbf{u} , we can solve equation, (11), which is again,

$$\int_K -\mathbf{q}_h \cdot \nabla v + \omega^2 u v \, d\mathbf{x} + \int_{\partial K} v \hat{\mathbf{q}}_h \cdot \mathbf{n}_K \, ds = \int_K f v \, d\mathbf{x}$$

We now separate $\hat{\mathbf{q}}$ into its independent parts of u_h and \mathbf{q}_h ; letting $\hat{\mathbf{q}} = \{\{\mathbf{q}\}\} + \beta[[\mathbf{q}]]$.

$$\int_K -\mathbf{q}_h \cdot \nabla v + \omega^2 u v \, d\mathbf{x} + \int_{\partial K} v \hat{\mathbf{q}}_h \cdot \mathbf{n}_K - v \tau[[u]] \cdot \mathbf{n}_K \, ds = \int_K f v \, d\mathbf{x}.$$

Finally, we can derive an equation solvable for u .

$$[J(-(M D_x)^T Q_x - (M D_y)^T Q_y + \omega^2 M) + B S J FluxQ - \tau B S J JumpU] \mathbf{u} = J M F,$$

where $FluxQ \mathbf{u} = [(\hat{\mathbf{q}} \cdot \mathbf{n}^-)_i]$ and $JumpU \mathbf{u} = [[u] \cdot \mathbf{n}^-]_i$ are vectors and $(\cdot)_i$ means evaluated at global degree of freedom, i .

5 Other DG Methods

Now that we have a clear understanding of the LDG method, we will examine most of the other DG methods from [1]. To understand and implement these DG methods, only the definitions of $\hat{\mathbf{q}}$ and \hat{u} need to be modified. To define these other DG methods we reproduce the table from [1], listing only the methods discussed here.

Method	\hat{u}	$\hat{\mathbf{q}}$
Bassi-Rebay	$\{\{u\}\}$	$\{\{\mathbf{q}\}\}$
LDG	$\{\{u\}\} - \beta \cdot [[u]]$	$\{\{\mathbf{q}\}\} + \beta [[\mathbf{q}]] - \tau [[u]]$
IP	$\{\{u\}\}$	$\{\{\nabla u\}\} - \tau [[u]]$
Baumann-Oden	$\{\{u\}\} + \mathbf{n}_K \cdot [[u]]$	$\{\{\nabla u\}\}$
NIPG	$\{\{u\}\} + \mathbf{n}_K \cdot [[u]]$	$\{\{\nabla u\}\} - \tau [[u]]$
Brezzi et al.	$\{\{u\}\}$	$\{\{\mathbf{q}\}\} - \tau \{\{r_e([u])\}\}$
Bassi et al.	$\{\{u\}\}$	$\{\{\nabla u\}\} - \tau \{\{r_e([u])\}\}$

Table 1: Selected DG Methods And Their Numerical Fluxes

We define $r_e(\cdot)$ weakly such that

$$(r_e([u]), \boldsymbol{\psi})_\Omega = -([u], \{\{\boldsymbol{\psi}\}\})_e, \forall \text{ edges, } e \in \mathcal{T}. \quad (39)$$

5.1 Non- $r_e(\cdot)$ Flux Methods

For the DG methods that do not contain $r_e(\cdot)$ in their flux terms, definition of the fluxes over the boundary is the only issue that needs to be resolved. Thereafter, their implementation becomes only a trivial change from the implementation of the LDG method. All of the flux calculations for these methods are still only averages and differences of nodal values across edges where the geometric locations of edge nodes on adjacent elements is the same. For the methods that include ∇ in their flux definition, a discrete differentiation matrix can be used. For example in matrix form,

$$\{\{\nabla u\}\} = 0.5 * (U^+ + U^-) D_x u_h. \quad (40)$$

A more subtle point that was never covered in any of the papers that I read, was whether

$$\mathbf{q} = \nabla u \text{ on } \Gamma. \quad (41)$$

Typically, but not always, I found that either case, equal or not equal, worked in my implementation.

We must make assumptions about u and \mathbf{q} on the boundary in order to define the fluxes there. This was a very frustrating endeavor for me, as this is not a well-defined process and some seemingly logical assumptions lead to convergent methods with non-symmetric operators for symmetric methods. For the two methods that should be symmetric, Bassi-Rebay and IP, we use assumptions from [2]. For the methods that inherently yield non-symmetric operators, Baumann-Oden and NIPG, we do a more “intuitive” boundary flux derivation and note that this type of derivation can lead to non-symmetric operators for the LDG, Bassi-Rebay and IP methods.

5.1.1 Other Symmetric DG Methods

The assumptions made in [2] are

$\hat{u} = g_{\mathcal{D}}$ on $\Gamma_{\mathcal{D}}$	$\hat{\mathbf{q}} = \mathbf{q}$ on $\Gamma_{\mathcal{D}}$
$\hat{u} = u_h$ on $\Gamma_{\mathcal{N}}$	$\hat{\mathbf{q}} = g_{\mathcal{N}}$ on $\Gamma_{\mathcal{N}}$

Table 2: Boundary Assumptions

These assumptions ensure a penalty-type enforcement of the boundaries, i.e. we have $g_{\mathcal{D}} - u^-$ and $g_{\mathcal{N}} - \mathbf{q}^-$ on the Dirichlet and Neumann boundaries, respectively. In summary, the boundary fluxes for these two methods are

The reader will notice that the $\hat{\mathbf{q}} - \mathbf{q}$ term on $\Gamma_{\mathcal{D}}$ for IP does not follow the assumptions in Table (2). If these assumptions are followed and this value is 0 as in Bassi-Rebay, the operator is non-symmetric although numerical experiments indicate that it is a convergent method. The IP formulation listed in Table (3) on $\Gamma_{\mathcal{D}}$ does yield a symmetric method. This formulation simply uses the interior definition of $\hat{\mathbf{q}}$ on $\Gamma_{\mathcal{D}}$.

		On $\Gamma_{\mathcal{D}}$	On $\Gamma_{\mathcal{N}}$
Bassi-Rebay	$\hat{u} - u$	$g_{\mathcal{D}} - u^{-}$	0
	$\hat{\mathbf{q}} - \mathbf{q}$	0	$g_{\mathcal{N}} - q^{-}$
IP	$\hat{u} - u$	$g_{\mathcal{D}} - u^{-}$	0
	$\hat{\mathbf{q}} - \mathbf{q}$	$\nabla u^{-} - \mathbf{q} - \tau(u^{-} + g_{\mathcal{D}})$	$g_{\mathcal{N}} - q^{-}$

Table 3: BassiRebay and IP Boundary Fluxes

It is interesting to note how the assumptions in Table (2) could yield contradictory conclusions for Bassi-Rebay. Letting $\hat{u} = g_{\mathcal{D}}$ implies that $u^{+} = u^{-}$ because $\hat{u} = \{\{u\}\}$. However, this implies a constant extension from the interior to the boundary, i.e. $q^{+} = 0$. Yet, q^{+} must equal q^{-} for $\hat{\mathbf{q}} - \mathbf{q} = 0$ to hold, because $\hat{\mathbf{q}} = \{\{\mathbf{q}\}\}$. $\mathbf{q}^{+} = \mathbf{q}^{-}$ implies a linear, not constant, extension. Perhaps, u^{+} and u^{-} are an “infinitely” small distance apart and can be considered equal. But sometimes on $\Gamma_{\mathcal{D}}$, $u^{+} = g_{\mathcal{D}}$ and $u^{+} \neq u^{-}$.

5.1.2 Non-Symmetric DG Methods

These methods inherently yield non-symmetric operators, so we don’t agonize over boundary fluxes in order to preserve symmetry. Instead, we just apply the interior definition of \hat{u} and $\hat{\mathbf{q}}$ to the boundary. The two methods covered here are Baumann-Oden and NIPG. We must make a few assumptions first, though.

$u^{+} = g_{\mathcal{D}}$ on $\Gamma_{\mathcal{D}}$	$\mathbf{q}^{-} = \mathbf{q}^{+}$ on $\Gamma_{\mathcal{D}}$ Or equivalently, $\nabla u^{-} = \nabla u^{+}$ on $\Gamma_{\mathcal{D}}$
$u^{-} = u^{+}$ on $\Gamma_{\mathcal{N}}$	$\nabla u = \mathbf{q}$ on $\Gamma_{\mathcal{D}}$ $\mathbf{q}^{+} = g_{\mathcal{N}}$ on $\Gamma_{\mathcal{N}}$ $\nabla u^{+} = g_{\mathcal{N}}$ on $\Gamma_{\mathcal{N}}$

Table 4: Boundary Assumptions 2

We first derive the boundary fluxes for Baumann-Oden, where

$\hat{u} = \{\{u\}\} + \mathbf{n}_{\mathbf{K}^-} \cdot \llbracket u \rrbracket$ and $\hat{\mathbf{q}} = \{\{\nabla u\}\}$.

$$\begin{aligned} \Gamma_{\mathcal{D}} : \hat{u} - u &= \frac{u^+ - u^-}{2} + \mathbf{n}_{\mathbf{K}^-} \cdot (u^+ \mathbf{n}_{\mathbf{K}^+} + u^- \mathbf{n}_{\mathbf{K}^-}) \\ &= \frac{u^+ - u^-}{2} + (u^- - u^+) \\ &= \frac{u^- - u^+}{2} = \frac{u^- - g_{\mathcal{D}}}{2} \end{aligned} \quad (42)$$

$$\Gamma_{\mathcal{D}} : \hat{\mathbf{q}} - \mathbf{q} = \nabla u - \mathbf{q} = 0 \quad (43)$$

$$\begin{aligned} \Gamma_{\mathcal{N}} : \hat{u} - u &= \frac{u^- - u^+}{2} + \mathbf{n}_{\mathbf{K}^-} \cdot (u^+ \mathbf{n}_{\mathbf{K}^+} + u^- \mathbf{n}_{\mathbf{K}^-}) \\ &= \frac{u^- - u^+}{2} = 0 \end{aligned} \quad (44)$$

$$\begin{aligned} \Gamma_{\mathcal{N}} : \hat{\mathbf{q}} - \mathbf{q} &= \frac{\nabla u^+ + \nabla u^-}{2} - \mathbf{q} \\ &= \frac{g_{\mathcal{N}}}{2} + \frac{\nabla u^-}{2} - \mathbf{q} \end{aligned} \quad (45)$$

If $\nabla u \neq \mathbf{q}$ on $\Gamma_{\mathcal{D}}$, computational problems were observed. The NIPG boundary fluxes are just a combination of previously defined fluxes, namely the NIPG flux is the same as Baumann-Oden on $\Gamma_{\mathcal{D}}$ and IP on $\Gamma_{\mathcal{N}}$. In summary, the boundary fluxes for these two methods are

		On $\Gamma_{\mathcal{D}}$	On $\Gamma_{\mathcal{N}}$
Baumann-Oden	$\hat{u} - u$	$\frac{u^- - g_{\mathcal{D}}}{2}$	0
	$\hat{\mathbf{q}} - \mathbf{q}$	0	$\frac{g_{\mathcal{N}}}{2} + \frac{\nabla u^-}{2} - \mathbf{q}$
NIPG	$\hat{u} - u$	$\frac{u^- - g_{\mathcal{D}}}{2}$	0
	$\hat{\mathbf{q}} - \mathbf{q}$	0	$g_{\mathcal{N}} - \mathbf{q}^-$

Table 5: Baumann-Oden And NIPG Boundary Fluxes

5.2 $r_{\epsilon}(\cdot)$ DG Methods

To properly understand how to implement the $r_{\epsilon}(\cdot)$ function, one needs to think of defining and implementing it weakly. These DG methods should yield symmetric operators.

5.2.1 Interior

We start by examining $r_e(\cdot)$ over the interior. Let $*^+$ and $*^-$ generically denote a function on adjacent elements for a shared edge.

$$\begin{aligned}
(r_e(\llbracket u \rrbracket), \boldsymbol{\psi})_\Omega &= -(\llbracket u \rrbracket, \{\!\!\{ \boldsymbol{\psi} \}\!\!\})_e, \forall \text{ edges } e \in \mathcal{T} \quad (46) \\
&\int_e -\frac{1}{2}(u^- \mathbf{n}^- + u^+ \mathbf{n}^+) \cdot (\boldsymbol{\psi}^+ + \boldsymbol{\psi}^-) ds \\
&\quad - \frac{1}{2} \int_e (-u^- \mathbf{n}^+ + u^+ \mathbf{n}^+) \cdot \boldsymbol{\psi}^+ ds - \frac{1}{2} \int_e (u^- \mathbf{n}^- - u^+ \mathbf{n}^-) \cdot \boldsymbol{\psi}^- ds \\
&\quad \frac{1}{2} \int_e (u^- \mathbf{n}^+ - u^+ \mathbf{n}^+) \cdot \boldsymbol{\psi}^+ ds + \frac{1}{2} \int_e (u^+ \mathbf{n}^- - u^- \mathbf{n}^-) \cdot \boldsymbol{\psi}^- ds \\
&\quad \frac{1}{2} ((u^- - u^+) \mathbf{n}^+, \boldsymbol{\psi}^+)_e + \frac{1}{2} ((u^+ - u^-) \mathbf{n}^-, \boldsymbol{\psi}^-)_e
\end{aligned}$$

Consider a basis for $\boldsymbol{\psi}$,

$$\text{span}(\boldsymbol{\psi}) = \text{span} \left(\begin{bmatrix} \psi_x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \psi_y \end{bmatrix} \right). \quad (47)$$

Then using $*_x$ to denote x components and not partial derivatives,

$$(r_e(\llbracket u \rrbracket)_x, \psi_x)_\Omega = \frac{1}{2} ((u^- - u^+) n_x^+, \psi_x^+)_e + \frac{1}{2} ((u^+ - u^-) n_x^-, \psi_x^-)_e. \quad (48)$$

A similar expression exists for the y -component.

We first calculate $r_e(\cdot)$ for a given edge and then calculate the contribution to the $\hat{\mathbf{q}}$ flux term in the final operator. We solve for $r_e(\cdot)$ in our finite element space. We find its projection onto the span of all the finite element basis functions. Using the matrix notation from the LDG implementation section, we get

$$JM \mathbf{r}_{e,x} = \frac{1}{2} BSJN_x (U^+ - U^-)|_e \mathbf{u} \quad (49)$$

$$\mathbf{r}_{e,x} = R_{e,x} \mathbf{u}, \quad (50)$$

where $\mathbf{r}_{e,x}$ is a vector, $R_{e,x}$ is a matrix and $(\cdot|_e)$ restricts the nonzero part of the range to the dofs on edge, e , for both K^+ and K^- . Similarly,

$$\mathbf{r}_{e,y} = R_{e,y} \mathbf{u}. \quad (51)$$

Now, we compute the contribution to the $\hat{\mathbf{q}}$ flux term involving $r_e(\cdot)$, $-\tau \{\!\!\{ r_e(\llbracket u \rrbracket) \}\!\!\}$. For a single element,

$$-\tau \sum_{e \in \partial K^-} (\{\!\!\{ r_e(\llbracket u \rrbracket) \}\!\!\}, \phi \mathbf{n}^-)_e. \quad (52)$$

In matrix format, this becomes for the x -component

$$-\sum_{e^- \in \partial K^-} \frac{\tau}{2} BSJN_x (U^+ + U^-)|_{e^-} R_{e,x} \mathbf{u}, \quad (53)$$

where $(\cdot|_{e^-})$ restricts the nonzero part of the range to the dofs living on the edge of K^- .

Here is one algorithm to calculate the contributions to the $\hat{\mathbf{q}}$ flux term that involve $r_e(\cdot)$. It is inefficient, but straightforward.

```

for k = 1:NumElements
  for e = 1:NumFaces
    % Calculate  $r_e$  on this face
     $e_1$  = dof's associated with  $e$  on  $\partial K^-$ 
     $e_2$  = dof's associated with  $e$  on  $\partial K^+$  and on  $\partial K^-$ 
     $\mathbf{r}_{e,x} = \frac{1}{2}(JM)^{-1}B SJ N_x (U^+ - U^-)|_{e_2} \mathbf{u}$ 
             =  $R_{e,x} \mathbf{u}$ 
     $\mathbf{r}_{e,y} = \frac{1}{2}(JM)^{-1}B SJ N_y (U^+ - U^-)|_{e_2} \mathbf{u}$ 
             =  $R_{e,y} \mathbf{u}$ 
    % Calculate Contribution to the  $\hat{\mathbf{q}}$  Flux Term and Store in  $R$ 
     $R = R - (\frac{\tau}{2}B SJ N_x (U^+ + U^-)|_{e_1} R_{e,x}) \mathbf{u}$ 
           -  $(\frac{\tau}{2}B SJ N_y (U^+ + U^-)|_{e_1} R_{e,y}) \mathbf{u}$ 
  end
end
% Add  $R_e$  to Operator Later

```

The non- $r_e(\cdot)$ flux terms should be analogous to previous methods

5.2.2 Boundary

We now examine the boundary contributions from the $r_e(\cdot)$ term. On $\Gamma_{\mathcal{N}}$, we let $u^+ = u^-$ which gives us

$$[[u]] = 0 \text{ on } \Gamma_{\mathcal{N}} \Rightarrow r_e([[u]]) = 0 \text{ on } \Gamma_{\mathcal{N}}. \quad (54)$$

On each edge of a Dirichlet boundary, $e_{\mathcal{D}} \in \Gamma_{\mathcal{D}}$,

$$(r_e([[u]]), \boldsymbol{\psi})_{\Omega} = -(u^- \mathbf{n}^-, \boldsymbol{\psi}^-)_{e_{\mathcal{D}}} + (g_{\mathcal{D}} \mathbf{n}^-, \boldsymbol{\psi}^-)_{e_{\mathcal{D}}}, \quad (55)$$

where we let $u^+ = g_{\mathcal{D}}$ and $\boldsymbol{\psi}^+ = \boldsymbol{\psi}^-$ in definition (39) so that we can combine terms. We solve for the x -components in matrix notation and note that the y -components are analogous.

$$J M \mathbf{r}_{e,x} = -B SJ N_x NtoFD|_{e_{\mathcal{D}}} \mathbf{u} + B SJ N_x NtoFD|_{e_{\mathcal{D}}} g_{\mathcal{D}} \quad (56)$$

$$\mathbf{r}_{e,x} = R_{e,x} \mathbf{u} + R_{e,x} g_{\mathcal{D}}, \quad (57)$$

where $(\cdot|_{e_{\mathcal{D}}})$ restricts the nonzero part of the range to the dofs on $e_{\mathcal{D}}$ and $NtoFD$ is a matrix similar to U^- except that the only nonzero part of its range is an injection onto $\Gamma_{\mathcal{D}}$. Now that $r_e(\cdot)$ can be calculated, the contribution to $\hat{\mathbf{q}}$ on $\Gamma_{\mathcal{D}}$ from the $r_e(\cdot)$ flux term, $-\tau \{r_e([[u]])\}$, can be found. For each $e_{\mathcal{D}}$, calculate

$$LHS = -\tau B SJ (N_x NtoFD|_{e_{\mathcal{D}}} R_{e,x} + N_y NtoFD|_{e_{\mathcal{D}}} R_{e,y}) \mathbf{u} \quad (58)$$

$$RHS = \tau B SJ (N_x NtoFD|_{e_{\mathcal{D}}} R_{e,x} + N_y NtoFD|_{e_{\mathcal{D}}} R_{e,y}) g_{\mathcal{D}} \quad (59)$$

The $g_{\mathcal{D}}$ term is constant and should be moved to the right-hand side, while the \mathbf{u} term should remain in the matrix equation on the left-hand side.

6 Multi-P Extension

Implementing different polynomial orders in adjacent elements is not very difficult. The only interesting implementation change is in defining U^+ to interpolate between adjacent elements with a different order. Previously in the constant order case, mapping values on a shared edge from one element to another included only using columns of the identity in U^+ because the geometric locations of nodes were equivalent for the two elements. A general algorithm for constructing U^+ is

```

for k = 1:NumElements
  for e = 1:NumFaces
    k2 = neighbor on (k, e)
    e1 = dof's associated with e on  $\partial K^-$ 
    e2 = dof's associated with e on  $\partial K^+$ 
    if ( $k_2 \notin \Gamma$ )
      if ( $p$  on  $k_2 == p$  on  $k$ )
         $U^+(e_1, e_2) = I$ 
      else
         $U^+(e_1, e_2) = P_{e_1, e_2}$ 
      end
    end
  end
end
end

```

All that remains is to define P_{e_1, e_2} . Lagrange interpolation formulae were used to interpolate between the two edges, but this yielded a non-symmetric operator, occasionally negative weights and interpolation aliasing. The operators generated this way did seem to give convergent methods, though.

The correct way to interpolate is to use the basis functions from the numerical implementation, i.e. Legendre basis functions. This avoids any aliasing, because when values are interpolated between edges, they are orthogonally projected into a higher or lower dimensional space. Let

$$P_{e_1, e_2} = V_{P_{e_1}} I_{P_{e_1}, P_{e_2}} (V_{P_{e_2}})^{-1}, \quad (60)$$

where $V_{P_{e_*}}$ is a Vandermonde matrix whose each column is a Legendre basis function evaluated at edge e_* 's nodes and $I_{P_{e_1}, P_{e_2}}$ is a rectangular "Identity" matrix of dimensions (num dofs on e_1) \times (num dofs on e_2). When P_{e_1, e_2} is applied to a vector on e_2 , $(V_{P_{e_2}})^{-1}$ will transform the nodal values to modal values. $I_{P_{e_1}, P_{e_2}}$ will then either truncate or fill in with 0's the vector of modal coefficients, depending on whether we are mapping from a lower to higher dimensional space or vice-versa. $V_{P_{e_1}}$ will then map the modal coefficients back

to nodal space. It should be again stressed that this suffers no aliasing because we are interpolating in modal space, i.e. we are either truncating or filling in with 0's the modal expansion of a function.

When building P_{e_1, e_2} or when just using columns of the Identity to construct U^+ , special care must be made for the case when the ordering of the nodes on the two edges are flipped. This can be a pesky error to track down.

7 Comments

1. It seems as if the literature concerning LDG methods has chosen β opportunistically so that desirable results for the fluxes are obtained. Most notably, the choice of β for boundaries seems to be only a matter of convenience so that desired outcomes occur.
2. The restrictions, (22) and (23), on \mathbf{q}^+ , \mathbf{q}^- , u^- , and u^+ over Γ give us well-defined boundary fluxes, but the restrictions seem to be only intuitive choices that lead to desirable outcomes.
3. How strong are Dirichlet boundary conditions imposed?
4. How do the strong-weak and weak forms differ? How do the spaces that each solution lives in affect things? How do these forms affect our solution on a computer?
5. For Bassi-Rebay, why choose linear extensions on boundaries by letting, $\mathbf{q}^+ = \mathbf{q}^-$? What would happen if constant extensions were done?
6. Generally, why is defining fluxes on the boundary such a kludge? Why is there not a general set of assumptions for u^+ , u^- , q^+ and q^- that work for all methods? Or, have I just not found it?

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