An Infinitesimal Approach to Calculus

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1 Introduction

This paper introduces the infinitesimal approach to calculus and then compares it with the $\epsilon - \delta$ approach. First, we will introduce the concept of infinitesimals and prove some important basic properties about them. Second, we will show how infinitesimals can be used to define differentiation and how this definition is equivalent to the $\epsilon - \delta$ definition of differentiation. Third, a similar presentation of using infinitesimals to define integration will be done, and this definition will be shown to be equivalent to the $\epsilon - \delta$ definition of integration.

The infinitesimal approach is the original and intuitive approach to calculus. The $\epsilon - \delta$ approach has become standard only because it was the first rigorous treatment of calculus. It was not until the 1960s that Abraham Robinson gave the first rigorous treatment of the infinitesimalapproach, and it is this approach which will be presented in this paper. It will hopefully be apparent to the reader that the infinitesimal approach is indeed simpler and more intuitive than the $\epsilon - \delta$ approach.

2 Infinitesimals

In this section, we will introduce the concept of infinitesimals and prove the necessary properties of infinitesimals for later sections. According to [1, p. 27], an **infinitesimal**, ϵ, is defined as

$$-a < \epsilon < a, \ \forall a \in \mathbb{R}, \ a > 0. \tag{1}$$

Clearly, the only real number that is also an infinitesimal is 0. Yet, does there exist at least one infinitesimal? This is, perhaps, the only "leap of faith" required for the infinitesimal approach to calculus. Like [1, p. 31], we assume the **Infinitesimal Axiom**.

There exists a positive infinitesimal number. (2)

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2.1 Constructing the Hyperreal Line

We will now construct the hyperreal line, $\mathbb{R}^*$, which is based on the real number line, $\mathbb{R}$. Intuitively, $\mathbb{R}^*$ is constructed by surrounding each real number, a, with infinitesimals. To do this, we first need a few rules from [1, pp. 31-33].

If ϵ is a positive infinitesimal, then $-\epsilon$ is a negative infinitesimal. (3)

If ϵ is a positive infinitesimal and $r \in \mathbb{R}$, then $r + \epsilon \in \mathbb{R}^*$, but $r \notin \mathbb{R}$. (4)

If ϵ is a positive infinitesimal and $a \in \mathbb{R}$ with $a > 0$, then $(a\ \epsilon)$ is also a positive infinitesimal. (5)

If ϵ is a positive infinitesimal, then $1/\epsilon$ is positive infinite and $-(1/\epsilon)$ is negative infinite. (6)

Positive infinite is defined to be a number that is greater than every real number and **negative infinite** is defined analogously. Proving these rules is straightforward, so we provide only the proofs of rules (5) and (6).
Proof of rule (5): let $r, a \in \mathbb{R},\ r, a > 0$ and ϵ be a positive infinitesimal, then

$$0 < \ \epsilon \ < \frac{r}{a} \Leftrightarrow 0 < a\ \epsilon < r. \tag{7}$$

Hence, $(a\ \epsilon)$ is less than any arbitrary real number, r.
Proof of rule (6): let $r \in \mathbb{R}$, $r > 0$ and ϵ be a positive infinitesimal, then

$$0 < \epsilon < \frac{1}{r} \Leftrightarrow \frac{1}{\epsilon} > r. \tag{8}$$

Hence, $\frac{1}{\epsilon}$ is greater than any arbitrary real number, r.

We now use these rules to create $\mathbb{R}^*$. Note that rule (5) gives us an infinite number of new infinitesimals as there are an infinite number of reals and that rule (3) gives us an infinite number of negative infinitesimals. Define the **hyperreal numbers**, $\mathbb{R}^*$, such that

$$\mathbb{R}^* = \{a + \epsilon : \ a \in \mathbb{R} \text{ and } \epsilon \text{ is an infinitesimal}\} \tag{9}$$

$$\bigcup \{1/\epsilon : \ \epsilon \text{ is an infinitesimal}\}. \tag{10}$$

It is interesting to note that there are no reals between a real number, a, and $a + \epsilon$, for all infinitesimals, ϵ. Also, the Archimedean property fails for hyperreals. If H is a positive infinite hyperreal, then there is no integer that is greater than H. Analagously to the real line, the commutative, associative, identity, inverse and distributive axioms hold for hyperreals. Also, the order axioms, the transitive law, sum law and product law, all hold for hyperreals. See [1, p. 30].

To complete our construction of the hyperreal numbers, we give two important theorems. Let ϵ and δ be infinitesimals, $b, c \in \mathbb{R}^*$ and H and K be infinite hyperreals. Theorem 1 from [1, p. 33] is

Negatives: (11)

$-\epsilon$ is infinitesimal. $-b$ is finite, but not infinitesimal. $-H$ is infinite.

Reciprocals:

If $\epsilon \neq 0$, $1/\epsilon$ is infinite. $1/b$ is finite, but not infinitesimal. $1/H$ is infinitesimal.

Sums:

$\epsilon + \delta$ is infinitesimal. $b + \epsilon$ is finite but not infinitesimal.

$b + c$ is finite and possibly infinitesimal. $H + \epsilon$ and $H + b$ are infinite.

Products:

$\delta\epsilon$ and $b\epsilon$ are infinitesimal. bc is finite but not infinitesimal.

Hb and HK are infinite.

Roots:

If $\epsilon > 0$, $\epsilon^{1/n}$ is infinitesimal. If $b > 0$, $b^{1/n}$ is finite but not infinitesimal.

If $H > 0$, $H^{1/n}$ is infinite.

The proof of this theorem is straightforward and some of the above rules have already been proved. Theorem 2 from [1, p. 37] is for any $b \in \mathbb{R}^*$,

(i) If b is between two infinitesimals, b is an infinitesimal. (12)

(ii) If b is between two finite hyperreal numbers, b is finite.

(iii) If b is greater than some positive infinite number, b is positive infinite.

(iv) If b is less than some negative infinite number, b is negative infinite.

The proof of this theorem is also straightforward, so we only show a proof of (iii). Let H, K be positive infinite and $H < K$ and $r \in \mathbb{R}$. Then $r < H \Rightarrow r < K$.

To illustrate computations with hyperreals, we now provide three examples from [1, pp. 35-37]. Let ϵ be a nonzero infinitesimal, $b, c \in \mathbb{R}^*$, and H be positive infinite. **Example 1:** $\frac{b-3\epsilon}{c+2\delta}$, is a finite hyperreal number divided by a finite hyperreal. Hence, it is a finite hyperreal. **Example 2:** $\frac{2H^2+H}{H^2-H+2} = \frac{2+1/H}{1-1/H+2/H^2}$, which is a finite hyperreal number divided by a finite hyperreal. Hence, it is a finite hyperreal. **Example 3:** $\frac{\epsilon^4+2\epsilon^2}{5\epsilon^4} = \frac{\epsilon^2+2}{5\epsilon^2}$, which is a finite hyperreal number divided by an infinitesimal. Hence, it is infinite.

Our construction of the hyperreal line is now complete.

2.2 Standard Parts

Define two hyperreal numbers, b and c, to be infinitely close if $b - c = \epsilon$, for some infinitesimal ϵ. We denote this with $b \approx c$. Note that the operator, $\approx$, is commutative and transitive.

Define the **standard part** of any $b \in \mathbb{R}^*$, $st(b)$, to be the real number to which b is infinitely close. Infinite hyperreals therefore have no standard part. $\forall b \in \mathbb{R}^*$, we have $st(b) \in \mathbb{R}$ and $b = st(b) + \epsilon$ for some infinitesimal, ϵ. We can derive some algebraic rules for $st(\cdot)$, which are given on [1, p. 40]. Let $a, b \in \mathbb{R}^*$ and finite.

$$st(b) \neq 0 \Rightarrow st\left(\frac{a}{b}\right) = \frac{st(a)}{st(b)} \tag{16}$$

$$st(-a) = -st(a). \tag{13}$$

$$st(a+b) = st(a) + st(b) \tag{14}$$

$$st(ab) = st(a)\ st(b) \tag{15}$$

$$st(a^n) = st(a)^n \tag{17}$$

$$a \geq 0 \Rightarrow st(a^{1/n}) = st(a)^{1/n} \tag{18}$$

$$a \leq b \Rightarrow st(a) \leq st(b) \tag{19}$$

Not all of the proofs of the above rules are straightforward, and the reader is directed to [1, pp. 41-42]. We provide the proof of (15) in order to indicate the overall nature of the proofs.

$$\begin{aligned} ab &= (r+\epsilon)(s+\delta), \text{ for } r, s \in \mathbb{R} \text{ and } \epsilon, \delta \text{ infinitesimals} \\ &= rs + r\delta + s\epsilon + \epsilon\delta \approx rs \Rightarrow st(ab) = rs. \end{aligned}$$

We end this section with three examples on how to take standard parts. Let $c \in \mathbb{R}^*$, $c = r + \epsilon$ and H be positive infinite, where $r \in \mathbb{R}$ and ϵ is an infinitesimal.

Example 1:

$$st\left(\frac{c^2+2c-24}{c^2-16}\right) = \frac{st(c^2)+st(2c)-st(24)}{st(c^2)-st(16)} = \frac{r^2+2r-24}{r^2-16} = \frac{r+6}{r+4}.$$

Example 2:

$$st\left(\frac{H^3+3H}{7H^3}\right) = \frac{st(1)+st(3/H^2)}{st(7)} = \frac{1}{7}.$$

Example 3:

$$st\left(\frac{3+\epsilon}{4\epsilon+\epsilon^2}\right) = \frac{st(3+\epsilon)}{st(4\epsilon+\epsilon^2)} = \frac{3}{0}, \text{which is undefined or has not standard part.}$$

The standard parts function is a critical component of working on the hyperreal line. It allows us to relate $\mathbb{R}^*$ to $\mathbb{R}$.

2.3 Function and Solution Axioms

The function and solution axioms are also critical components of working on the hyperreal line, as they allow us to relate functions with $\mathbb{R}^*$ as their domain to functions with $\mathbb{R}$ as their domain and vice versa. A **hyperreal function** is defined analogously to real valued functions, in that a hyperreal function is just a set of ordered pairs of hyperreal numbers. More precisely, for any hyperreal function, f, and $\forall a \in \mathbb{R}^*$, either there is only one hyperreal number, b, such that $f(a) = b$ or f is undefined at a. This leads us to the **function axiom** on [1, p. 65].

$$\text{For every real function, } f\text{, there is a corresponding hyperreal function, } f^*\text{, of the same number of variables, called the natural extension of } f. \tag{20}$$

To complete the connection between f and f^*, we need to connect the ranges of the two functions. To define f^* over $\mathbb{R}^*$, we use the **solution axiom** on [1, p. 65].

$$\text{If two systems of formulas have exactly the same real solutions, then they have exactly the same hyperreal solutions.} \tag{21}$$

As a consequence of this axiom, f^* will behave analogously to f.

1. If $r \in \mathbb{R}$ and $f(r)$ exists, then $f^*(r) = f(r)$.
2. If $r \in \mathbb{R}$ and $f(r)$ is undefined, then $f^*(r)$ is undefined.
3. If a real function, f, is defined by $f(x) = T(x)$, where $T(x)$ is a term involving x, then the natural extension of f is given by $f^*(x) = T(x), \forall x \in \mathbb{R}^*$.

One can extend functions such as $f(x) = \sqrt{1-x^2}, \forall x \in \mathbb{R}$ to $f^*(x) = \sqrt{1-x^2}, \forall x \in \mathbb{R}^*$. However, not every hyperreal function is the natural extension of a real function. For instance, the standard parts function is not the natural extension of any real function, including the identity function, $x = f(x)$. This can be seen on [1, p. 68], where Keisler considers intervals of infinitely small length on $\mathbb{R}^*$ that contain only one real number. On these intervals, the standard parts function is constant, while the natural extension of $x = f(x)$ is at a 45^o angle.

3 Differentiation

We now use the tools for working on the hyperreal line built up in the previous section to define differentiation. Then, equivalence of the infinitesimal definition of differentiation to the $\epsilon - \delta$ definition of differentiation will be shown.

3.1 Using Infinitesimals to Define Differentiation

The derivative of a function, $f(x)$, at a, is the slope of f at the point, a. This is a little problematic, as slope means rise over run. Yet if we want the slope at a single point, how do we define the run? We define the run using infinitesimals, i.e. we consider only an infinitesimally small run. According to [1, p. 72] the **slope**, S, of a function f at $a \in \mathbb{R}^*$ is $S = st\left(\frac{f(a+\Delta x)-f(a)}{\Delta x}\right)$, where Δx is any nonzero infinitesimal. The slope of f at a exists if and only if S is finite and has the same standard part for all nonzero infinitesimals, Δx.

The **derivative** of f is f' and is defined to be a function such that $f'(a)$ is the slope of f at a. Hence,

$$f'(x) = st\left(\frac{f(x+\Delta x) - f(x)}{\Delta x}\right) = st\left(\frac{\Delta y}{\Delta x}\right), \tag{22}$$

$\forall x \in \mathbb{R}^*$ such that the slope of f exists at x and that $f'(x)$ is the same for all nonzero infinitesimals, Δx. In order to simplify our notation, define a new dependent variable, $\Delta y = f(x+\Delta x) - f(x)$.

We now give two examples from [1, pp. 74-78] on finding the derivative.

Example 1: Let $f(x) = y = x^3$.

$$\begin{aligned} y + \Delta y &= (x+\Delta x)^3 \\ \frac{\Delta y}{\Delta x} &= \frac{(x+\Delta x)^3 - x^3}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2 \\ &\text{Taking standard parts, } f'(x) = 3x^2. \end{aligned}$$

Example 2: Let $f(x) = y = \sqrt{(}x)$.

Case 1: $x < 0$. $f(x)$ is undefined for $x < 0$, so $f'(x)$ is undefined.

Case 2: $x = 0$. $f(0+\Delta x)$ is undefined for $\Delta x < 0$, so $f'(0)$ is undefined.

Case 3: $x > 0$. $y + \Delta y = \sqrt{x+\Delta x}$

$$\frac{\Delta y}{\Delta x} = \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} = \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}\,\frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$\frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}}$$

Take standard parts to find the derivative.

$$st\left(\frac{\Delta y}{\Delta x}\right) = \frac{st(1)}{st(\sqrt{x+\Delta x}) + st(\sqrt{x})} = \frac{1}{2\sqrt{x}}$$

One last important theorem is the **increment theorem** on [1, p. 83].

> Let $f'(x)$ exist for any $x \in \mathbb{R}^*$, then Δy is an infinitesimal. Namely, (23)
> $\Delta y = f'(x)\Delta x + \epsilon\ \Delta x$,
> for some infinitesimals Δx and ϵ, where ϵ is a function of x and Δx.

The proof of this theorem is straightforward; it is essentially an infinitesimal version of a first-order Taylor expansion. If $\Delta x = 0$, then $\Delta y = \epsilon = 0$. Otherwise, define ϵ to be,

$$\epsilon = \frac{\Delta y}{\Delta x} - f'(x) \ \Leftrightarrow\ \Delta y = f'(x)\Delta x + \epsilon\Delta x.$$

We close this section by deriving some familiar differentiation rules. The **sum rule** from [1, p. 87] is

$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}, \tag{24}$$

where familiarity with the notation $du/dx = u'(x)$ is assumed. Let $y = u + v$ and Δx be an infinitesimal.

$$\begin{aligned} y + \Delta y &= (u + \Delta u) + (v + \Delta v) \\ \Delta y &= (u + \Delta u) + (v + \Delta v) - (u + v) \\ \frac{\Delta y}{\Delta x} &= \frac{\Delta u + \Delta v}{\Delta x} \end{aligned}$$

Take standard parts.

$$st\left(\frac{\Delta y}{\Delta x}\right) = st\left(\frac{\Delta u + \Delta v}{\Delta x}\right) \quad \Leftrightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

The **chain rule** from [1, p. 104] is

> Let f, G be two hyperreal functions and define $g(t) = G(f(t))$. (25)
> Wherever $f'(t)$ and $G'(f(t))$ exist, $g'(t) = G'(f(t))\ f'(t)$.

Let $x = f(t), y = g(t), y = G(x)$ and $\Delta t \neq 0$ be an infinitesimal. t is the independent variable. Using the increment theorem on $x = f(t)$ and $y = G(x)$, we get infinitesimal increments Δx and Δy. More precisely,

$$\Delta y = G'(x)\Delta x + \epsilon \Delta x \quad \Leftrightarrow \quad \frac{\Delta y}{\Delta t} = G'(x)\frac{\Delta x}{\Delta t} + \epsilon \frac{\Delta x}{\Delta t}$$

Take standard parts.

$$\begin{aligned} st\left(\frac{\Delta y}{\Delta t}\right) &= st\left(G'(x)\frac{\Delta x}{\Delta t}\right) + st\left(\epsilon \frac{\Delta x}{\Delta t}\right) \\ g'(t) &= G'(x)f'(t) = G'(f(t))f'(t). \end{aligned}$$

3.2 Evaluating Limits

In order to define differentiation with the $\epsilon - \delta$ definition, we first must define how to evaluate limits. We assume familiarity with the $\epsilon - \delta$ approach to calculus and simply restate some definitions. According to [2, p. 152], the $\epsilon - \delta$ **definition of a limit** is

> Let f be a function on $J \backslash \{a\}$ for some open interval, J, containing a and let L $\in \mathbb{R}$. Then $lim_{x \to a} f(x) = L \Leftrightarrow \forall \epsilon > 0, \exists \delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$. (26)

Intuitively when x is close to a, the function's values should also be close to the limit, but it is hard to apply this definition rigorously. The infinitesimal approach to limits is more intuitive and easier to apply. According to [1, p. 299], the **infinitesimal definition of a limit** is

$$lim_{x \to a} f(x) = L \tag{27}$$

means that whenever a hyperreal number x is infinitely close to but not equal to a, $f(x)$ is infinitely close to L.

This definition has a similar intuition. Taking the limit of an expression means evaluate infinitely close to the limit point and then ignore infinitely small parts of the result.

According to [1, p. 303], these two definitions are equivalent. We will abbreviate the proof here for reasons of space. Assume that the $\epsilon - \delta$ condition holds, then we must show that for a hyperreal number, x, $x \approx a \Rightarrow f(x) \approx L$. Equivalently, we can show that $|f(x) - L| < \epsilon, \forall \epsilon \in \mathbb{R}$. Given the $\epsilon - \delta$ condition, we know that for any $\epsilon > 0$, there is a corresponding, $\delta \in \mathbb{R}$. Any number, $x \in \mathbb{R}^*$, infinitely close to a is certainly within δ of a, so by the solution axiom, $|f(x) - L| < \epsilon$. But this holds for any $\epsilon \in \mathbb{R}$, so we are done. To prove the other way, we do it by contradiction. Assume that the $\epsilon - \delta$ condition is false for some $\epsilon > 0$, but that the infinitesimal definition of limits still holds. Take the ϵ that fails the $\epsilon - \delta$ condition for limits. Then $\forall \delta > 0, |a - x| < \delta \Rightarrow |f(x) - L| > \epsilon$. Applying the solution axiom, we see this holds also $\forall x^* \in \mathbb{R}^*$, so that $|f(x^*) - L| > \epsilon$, which is a contradiction.

3.3 Equivalence of the Two Definitions of Differentiation

We assume familiarity with the $\epsilon - \delta$ defn of differentiation given in class and simply repeat the definition from [2, p. 205]. Let f be a function defined on some open interval about a, then the $\epsilon - \delta$ **definition of differentiation** is

$$f'(a) = lim_{x \to a} \frac{f(x) - f(a)}{x - a} = lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}. \tag{28}$$

Remember the infinitesimal definition of differentiation, equation (22), $f'(x) = st\left(\frac{\Delta y}{\Delta x}\right)$. According to definition (27), evaluating the limit of a function, f, at a point, a, with infinitesimals means simply take the standard part of the function evaluated at any value that is infinitely close to a. Hence, we can conclude that both definitions of differentiation are equivalent to

$$f'(a) = lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}. \tag{29}$$

The only difference, then, between the infinitesimal definition of differentiation and the $\epsilon - \delta$ definition of differentiation is how you evaluate this limit. Yet, both methods of limit evaluation are equivalent, so both definitions of differentiation are equivalent.

4 Integration

We now use the tools for working on the hyperreal line to define the definite integral. It will also be shown that the infinitesimal definition of a definite integral is equivalent to the $\epsilon - \delta$ definition of the definite integral.

Before we begin our discussion of integration, we must define partitions on the hyperreal line. Using the simplified definition of partitions on [1, p. 155], define a **partition**, P, of $[a, b] \in \mathbb{R}^*$, to be formed by choosing a positive integer, n, and dividing $[a, b]$ into n equal parts, with the last subinterval possibly being shorter than the rest.

Define hyperrintegers to be infinite hyperreals that correspond to the natural extension of the set of integers in $\mathbb{R}$. See [1, p. 157-9]. What happens when n is a positive infinite hyperrinteger? Our interval is divided into infinitely many small parts, each of an infinitesimal length. The length of each subinterval is $\Delta x = (b - a)/n$. Such a partition is called an **infinite partition**. Therefore, every hyperreal number on $[a, b]$ is infinitely close to a partition point.

4.1 Using Infinitesimals to Define Integration

In this section, we restrict, as in [1], our attention to a subset of integrable functions by assuming that f is continuous on a fixed and closed interval, $I = [a, b] \in \mathbb{R}^*$.

We define an **area function** as on [1, p. 218]. An area function for f and $c, d \in [a, b]$ is denoted by $A(c, d)$ and has the addition property, rectangle property, $A(a, a) = 0$, and $A(b, a) = -A(a, b)$. The **addition property** states that $A(a, b) = A(a, c) + A(c, b), \forall c \in I$. The **rectangle property** states that for $m = min(f, [a, b]), M = max(f, [a, b]),\ m(b - a) \leq A(a, b) \leq M(b - a)$.

Define a **Riemann sum**, $R(P)$, to correspond to a partition, P, of I restricted to $\mathbb{R}$. Let Δx be the uniform length between any two partition points, i and $i - 1$, and $f(x_i)$ be f evaluated at the *i-th* partition point.

$$R(P) = \Sigma_a^b f(x_i)\Delta x. \tag{30}$$

A natural question is what happens to the Riemann sum when the partition is infinite? We let $\widetilde{P}$ be such an infinite partition over $[a, b]$ with uniform infinitesimal spacing $\widetilde{\Delta x}$. $R(P)$ is strictly a function of Δx as long as I is held fixed. Applying the solution axiom, the Riemann sum can also become a function of infinitesimals, so that the infinite Riemann sum is defined to be,

$$\widetilde{R}(\widetilde{P}) = \Sigma_a^b f(x_i)\widetilde{\Delta x}. \tag{31}$$

Next, we prove that the infinite Riemann sum is a finite hyperreal number. Using f's continuity, f must take a maximum and minimum on I.

Let $B = max(|f|, [a, b])$ and $C = min(|f|, [a, b])$. Then, (32)

$C(b - a) \leq \widetilde{R}(\widetilde{P}) \leq B(b - a)$. Hence, the infinite Riemann sum is finite.

Define the **definite integral** of f on I with $\widetilde{\Delta x} > 0$ an infinitesimal to be the standard part of the infinite Riemann sum.

$$\int_a^b f(x)dx = st(\widetilde{R}(\widetilde{P})), \quad \int_a^a f(x)dx = 0, \quad \int_b^a f(x)dx = -\int_a^b f(x)dx. \tag{33}$$

We now prove some familiar integration rules. The **sum rule**,

$$\int_a^b f(x) + g(x)\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx. \tag{34}$$

Consider the finite Riemann sums,

$$\Sigma_a^b(f(x_i) + g(x_i))\Delta x = \Sigma_a^b f(x_i)\Delta x + \Sigma_a^b g(x_i)\Delta x \tag{35}$$

By the solution axiom, this holds for $\widetilde{\Delta x}$ an infinitesimal. Therefore, the definite integrals of the summations are equal.

The proofs of the **inequality rule** and the **constant rule**,

$$f(x) \le g(x) \text{ on } [a,b] \Rightarrow \int_a^b f(x)\ dx \le \int_a^b g(x)\ dx, \tag{36}$$

$$\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx, \text{ for } c \text{ constant}, \tag{37}$$

are analogous. One first shows that the rule holds for finite Riemann sums. Then one applies the solution axiom to show that the same holds for infinitesimals.

Two natural questions arise; is the definite integral the same for different infinitesimals? and, is equation (33) an area function? Both statements are true, which implies that **the definite integral is the unique area function of f**.

We follow the proof on [1, pp.235-6] that the definite integral using two different positive infinitesimals, $\widetilde{\Delta x_1}, \widetilde{\Delta x_2}$, is the same. Let $r \in \mathbb{R}$ and $c = r/(b-a)$ and construct two infinite Riemann sums, $\Sigma_a^b f(x_i)\widetilde{\Delta x_1}$ and $\Sigma_a^b(f(x_i)+c)\widetilde{\Delta x_2}$.

It follows from $c > 0$ that, $\Sigma_a^b f(x_i)\widetilde{\Delta x_1} \le \Sigma_a^b(f(x_i) + c)\widetilde{\Delta x_2}$. (38)

Take standard parts: $\displaystyle\int_a^b f(x)dx_1 \le \int_a^b (f(x) + c)dx_2$.

Use the sum rule: $\displaystyle\int_a^b f(x)dx_1 \le \int_a^b f(x)dx_2 + \int_a^b c\ dx_2$

$$\int_a^b f(x)dx_1 \le \int_a^b f(x)dx_2 + c(b-a) = \int_a^b f(x)dx_2 + r.$$

This holds $\forall r > 0, r \in \mathbb{R}, \Rightarrow \displaystyle\int_a^b f(x)dx_1 \le \int_a^b f(x)dx_2$.

A similar argument shows the opposite inequality. Hence, the definite integrals for each infinitesimal are equal. As a corollary, the definite integral is unique. Note that dx_1 and dx_2 refer to definite integrals with different infinitesimals.

The definite integral in (33) has all four properties of an area function.

Rectangle Property: Let $B = max(|f|, [a, b])$ and $C = min(|f|, [a, b])$. (39)

$$m \leq f(x) \leq M \Leftrightarrow \int_a^b m\ dx \leq \int_a^b f(x)\ dx \leq \int_a^b M\ dx$$

$$\Leftrightarrow m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a),$$

by applying the inequality rule.

Addition Property: Considering finite Riemann sums for $c \in [a, b]$, (40)

$\Sigma_a^b f(x_i)\Delta x = \Sigma_a^c f(x_i)\Delta x + \Sigma_c^b f(x_i)\Delta x$.

By the solution axiom, this holds for $\widetilde{\Delta x}$ an infinitesimal.

Therefore, the definite integrals of the summations are equal.

By construction, the definite integral satisfies the other two properties of an area function, $A(a, a) = 0$, and $A(b, a) = -A(a, b)$. Hence, the definite integral is an area function. [1, pp. 237 - 8]

4.2 Equivalence of the Two Definitions of Integration

We now prove that the definite integral defined in (33) is equivalent to the one studied in class. Familiarity with the definition given in class is assumed, and we simply restate the $\epsilon - \delta$ **definition of integration** from [2, p. 248].

A bounded function f on $[a, b]$ is integrable if and only if $\forall\epsilon > 0, \epsilon \in \mathbb{R}$, (41)
there exists a partition, P, of $[a, b] \in \mathbb{R}$ such that $U(f, P) - L(f, P) < \epsilon$.
Then, the definite integral of f on $[a, b]$ is between $U(f, P)$ and $L(f, P)$
for all partitions of $[a, b]$.

$U(f, P)$ and $L(f, P)$ refer to upper and lower sums as discussed in class.

To show equivalence, we use f's continuity to assume that the integral in definition, (41), exists. This property was proved in class. We also remember proof (32) that uses f's continuity to guarantee the existence of the integral defined with infinitesimals in equation, (33). Equivalence is shown by proving that any infinite Riemann Sum is between any $U(f, P)$ and $L(f, P)$, where P is a partition of $I \in \mathbb{R}$. Recall the statement proved in class that as you refine a partition on the real line, the upper sums go down and the lower sums go up. We extend this fact to the hyperreal line by applying the solution axiom. Therefore, the infinite Riemann sum of any infinite partition is between $U(f, P)$ and $L(f, P)$, for any partition, P, of $I \in \mathbb{R}$. Hence, we can restrict the infinite Riemann sum to be inside any interval of arbitrary length with the integral from definition (41). It follows that both definitions of integration are equivalent.

5 Conclusion

1. Using infinitesimals seems to be more intuitive than the $\epsilon-\delta$ method used in class. The proofs tend to be shorter and easier to follow.

2. Infinitesimals and the $\epsilon - \delta$ method yield equivalent definitions of differentiation and integration.

3. Standard analysis is built on evaluating limits with the $\epsilon - \delta$ method. Replacing this with the infinitesimal method of evaluating limits leads to an entirely different way of doing analysis.

4. The only "leap of faith" for the infinitesimal approach to Calculus is the infinitesimal axiom. [1] provides no motivation for this assertion and simply leaves the reader wondering. Showing a definitive proof that one infinitesimal exists is needed to make this approach to analysis rigorous. One could possibly define a positive infinitesimal to be a positive real number that is less than any positive real number that one could enumerate in a finite amount of time. This avoids the seeming contradiction to the density of reals because there is no real number in the interval of nonzero length between 0 and any infinitesimal.

References

[1] H. J. Keisler, *Elementary Calculus*, Prindle, Weber and Schmidt, Boston, 1976.

[2] K. A. Ross, *Elementary Analysis: The Theory of Calculus*, Springer, New York, 1980.