Jacob Schroder

UIUC

The Search for a General Strength of Connection Measure in AMG

CS 550

Outline

Say Goodbye to M-Matrices

In an ideal world, AMG should only need a matrix and a smoother, i.e. be a black box solver. That in turn implies that we want a black box strength of connection measure.

The classic Ruge-Stüben, R-S, strength measure covered in class works only for M-matrices. All kinds of problems can generate non M-matrices, such as relatively simple ones like the bi-harmonic or certain anisotropies.

We need better strength of connection information.

Strength Is Important

- Good C-F splitting
- Aggregation done well
- Nonzero placement in the interpolation operator

Algebraically smooth error is the error not reduced by smoothing, the space complementary to the smoother.

$$S = (I - \alpha D^{-1} A).$$

$Se \approx e$ defines smooth error, where

$$\alpha = \frac{K}{\rho(D^{-1}A)} \text{ and } \rho(S) \approx 1.$$

The high energy eigenvectors of S define the smooth error. This space corresponds to the low energy eigenvectors of A.

Strength of Connection

- A strong connection between degrees of freedom i and j implies that we can reliably interpolate smooth error between them.
- The chicken and the egg – Defining strong connections assumes some interpolation method, and constructing interpolation operators assumes a definition of smooth error!
- We will concern ourselves only with strength of connection. However, we can say that we want our interpolation basis functions to have low energy, i.e. have a small energy norm (smoothed aggregation). Hence, we may want to define strength using the energy norm.

Look to A^{-1}

The eigenspace of S defines smooth error, but eigen-calculations are expensive, and it is not clear for an arbitrary operator where to truncate the eigenspace when capturing smooth error.

Can we think of a more accessible heuristic? A^{-1} relates the size of the residual at dof, j, to the size of the error at dof, i.

$$Ae = r \Leftrightarrow A^{-1}r = e. \tag{1}$$

This is the equation MG solves.

Construct a strength of connection matrix, $\widetilde{A}$, by defining the strength of connection between dof, i, and dof, j, as

$$\widetilde{A}_{ij} = \frac{\| \, G_{:,i} - (\, (G_{j,i}) \, I_{:,j} \,) \, \|_A}{\| \, G_{:,i} \, \|_A}, \tag{2}$$

where $I_{:,j}$ is the j-th column of the identity and $G_{:,i}$ is the i-th column of A^{-1} as approximated by some local relaxation scheme such as weighted-Jacobi.

Compare values in $\widetilde{A}_{i,:}$ in order to determine the strong neighbors of dof, i.

What Is It Actually Measuring?

- Did the removal of the point from $G_{:,i}$ create large derivatives?
- If dof, i, were the only coarse grid point, then a good interpolation operator would be column i of the inverse (think small energy norm for good basis functions).

  If the removal of point j from $G_{:,i}$ creates large values in the energy-norm, then points i and j are probably strongly connected, because point j's removal greatly disrupted the low energy of $G_{:,i}$.

  Find those points in $G_{:,i}$ most important to maintaining its *minimal* energy.

All results are for the Poisson problem on a regular 33×33 grid of bilinear quadrilaterals. Matrices used were non M-matrices. $\mu =$ the number of weighted-Jacobi steps.

μ	Stencil		
3	0.0006	0.0641	0.0245
	0.0028	0.0000	0.0028
	0.0245	0.0641	0.0006
4	0.0012	0.0852	0.0316
	0.0018	0.0000	0.0018
	0.0316	0.0852	0.0012

Table: $\widetilde{A}$ stencil at point (17,8)

Results - Stencils in $\widetilde{A}$
Rotated Anisotropy of $\frac{\pi}{4}$

μ	Stencil		
2	0.0011	0.0073	0.0378
	0.0073	0.0000	0.0073
	0.0378	0.0073	0.0011
3	0.0012	0.0151	0.0634
	0.0151	0.0000	0.0151
	0.0634	0.0151	0.0012
4	0.0010	0.0230	0.0840
	0.0230	0.0000	0.0230
	0.0840	0.0230	0.0010

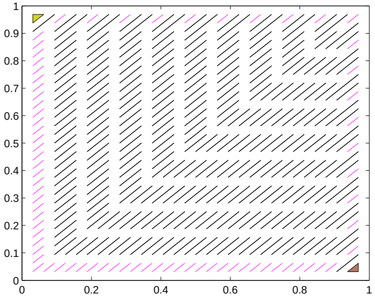
Table: $\widetilde{A}$ stencil at point (17,8).

Results - Stencils in $\widetilde{A}$
Coefficient Jump In Center of Domain

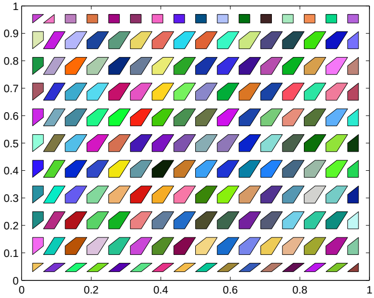
μ	Point	Stencil		
4	(17,8)	0.0356	0.0240	0.0004
		0.0469	0.0000	0.0005
		0.0356	0.0240	0.0004
4	(18,8)	0.0003	0.0149	0.0160
		0.0005	0.0000	0.0211
		0.0003	0.0149	0.0160

Table: $\widetilde{A}$ stencils around interface.

Results - Sample Aggregates
Hot Dog!



Rotated Anisotropy of $\frac{\pi}{4}$



Rotated Anisotropy of $\frac{\pi}{8}$

- The core idea is simple and intuitive; look at the entries of A^{-1} to determine strength of connection.
- Effective for at least some non M-matrices, unlike the classic strength measure.
- Produced MG preconditioners for CG that were as effective as using distance-based strength of connection measures.
- However, the measure is not without its drawbacks.

What happens when the approximate column of the inverse approaches the actual column of the inverse?

Stencil		
0.1282	0.4390	0.6246
0.3854	0.0000	0.3800
0.4812	0.3162	0.0712

Table: $\widetilde{A}$ stencil at point (17,8) – Rotated Anisotropy of $\frac{\pi}{4}$

Perhaps, A^{-1} with its unrestricted stencil smears information. Use SPAI [4] to generate the approximate inverse, G. Enforce $\text{nnz}(G) \approx \text{nnz}(A)$.

Stencil		
0.0051	0.1450	0.0042
0.0095	0.0000	0.0079
0.0057	0.1599	0.0039

Table: $\widetilde{A}$ stencil at point (17,8) – Vertically strong anisotropy

SPAI offers limited improvement over direct usage of A^{-1}.

Stencil		
0.0000	0.0374	0.0921
0.0378	0.0000	0.0285
0.0944	0.0288	0.0000

Table: $\widetilde{A}$ stencil at point (17,8) – $\frac{\pi}{4}$ rotated anisotropy

Stencil		
0.0040	0.0997	0.0227
0.0005	0.0000	0.0005
0.0227	0.0997	0.0040

Table: $\widetilde{A}$ stencil at point (17,8) – $\frac{\pi}{8}$ rotated anisotropy

Problem 1 - Cost

k Jacobi relaxations with a 0 initial guess to calculate G is equivalent to doing $k-1$ mat-mats with A. Calculating the measure increases the cost to k mat-mats.

Problem 2 - Not a Panacea

The measure did not perform well for the most difficult test case, a stretched and unstructured grid on an oval-shaped domain. Yet, it did do as well as a distance-based strength of connection measure for this problem.

Problem 3 – The Real Problem

- A^{-1} relates degrees of freedom with respect to all error for the system, $Ae = r$. This includes smooth error, but also error effectively reduced by smoothing.
- Ideally, we want to know for any dof, j, on the fine grid, what are the most important coarse dof's for interpolating smooth error to j?
- This issue is partially avoided by the fact we only relax a handful of times when calculating the approximate inverse. After only a few iterations of relaxation, the approximation to the inverse should still be heavily influenced by the relaxation method and hence, hopefully, local and reflective of the relaxation method.

Conclusions

- We need accurate strength of connection information to form coarse grids and interpolation operators in multigrid.
- The proposed measure in [2] is effective for a more general class of matrices than the classic strength of connection measure.
- However, the proposed measure is not without its drawbacks.
  - The cost casts doubt on the computational feasibility of this measure.
  - What happens to the measure in the limit, when using actual columns of A^{-1}, casts doubt on the idealness of the measure.

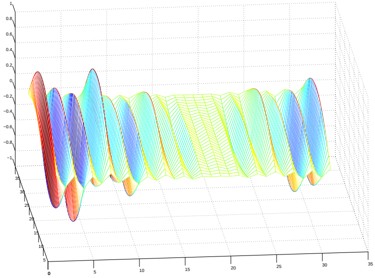
Future Work

- Reduce the cost of the measure by using some further approximation to it or by only calculating for rows that deviate strongly from being "M" like.
- Explore adaptive threshholding by varying the number of weighted-Jacobi iterations.
- Explore other strength measures that are based on approximating the smooth components of S's eigenspace.

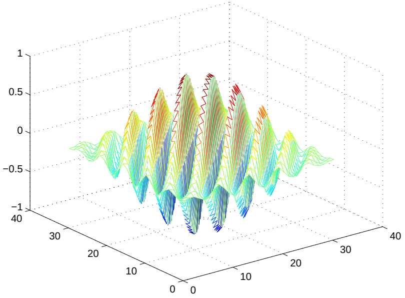
1. A. Brandt, S. McCormick and J. W. Ruge. *Algebraic multigrid (AMG) for sparse matrix equations.* Sparsity and Its Applications, DJ Evans, ed., Cambridge University Press, Cambridge, 1984.
2. J. Brannick, M. Brezina, S. MacLachlan, T. Manteuffel, S. McCormick, and J. Ruge. *An energy-based AMG coarsening strategy.* Num. Lin. Alg. Appl., 2006: **13**:133-148.
3. J. W. Ruge and K. Stüben. *Algebraic Multigrid,* in Multigrid Methods. S. McCormick, ed., SIAM, Philadelphia, 1987, pp. 73-130.
4. M. Grote and T. Huckle. *Parallel preconditioning with sparse approximate inverses.* SIAM J. Sci. Comput., 1997: **18**:838-853.

Dominant Eigenvectors of the Fake Cycle Matrix

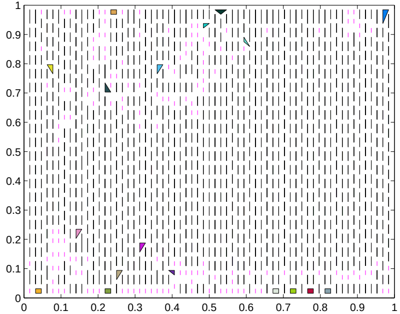
Fake Cycle Matrix $:= (I - P^T P)S$



Vertically Strong Anisotropy



Rotated by $\frac{\pi}{4}$ Anisotropy



Resulting aggregation using eigenvector

$$g_0 = 0 \tag{3}$$
$$g_1 = 0 + \omega D^{-1}(e_j - 0) = \omega D^{-1} e_j$$
$$g_2 = \omega D^{-1} e_j + \omega D^{-1}(e_j - A\omega D^{-1} e_j)$$
Which is $O(D^{-1}AD^{-1}e_j)$.

Considering just this dominant operation, the following pattern becomes clear.

$$g_3 = O(D^{-1}AD^{-1}AD^{-1}e_j) = O(A^2 e_j) \tag{4}$$
$$g_4 = O(D^{-1}A(D^{-1}AD^{-1}AD^{-1}e_j) = O(A^3 e_j)$$

What if off-diagonals become positive or the matrix loses its definiteness? The classic strength of connection measure begins to break down. For example, the rotated anisotropic diffusion problem,

$$-(\nu c^2 + s^2)u_{xx} + 2(1-\nu)sc\, u_{xy} - (\nu s^2 + c^2)u_{yy} = f, \qquad (5)$$

with $\nu = 0.001, c = cos(\frac{\pi}{8})$ and $s = sin(\frac{\pi}{8})$ yields a matrix stencil of

$$\begin{matrix} 0.0098 & -0.5200 & -0.3434 \\ 0.1864 & 1.3347 & 0.1864 \\ -0.3434 & -0.5200 & 0.0098 \end{matrix},$$

for bilinear quads on a structured grid.